Competing Orders in One-Dimensional Half-Integer Fermionic Cold Atoms: A Conformal Field Theory Approach

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The physical properties of arbitrary half-integer spins $F = N - 1/2$ fermionic cold atoms loaded into a one-dimensional optical lattice are investigated by means of a conformal field theory approach. We show that for attractive interactions two different superfluid phases emerge for $F \geq 3/2$: A BCS pairing phase, and a molecular superfluid phase which is formed from bound-states made of $2N$ fermions. In the low-energy approach, the competition between these instabilities and charge-density waves is described in terms of $\mathbb{Z}_N$ parafermionic degrees of freedom. The quantum phase transition for $F = 3/2, 5/2$ is universal and shown to belong to the Ising and three-state Potts universality classes respectively. In contrast, for $F \geq 7/2$, the transition is non-universal. For a filling of one atom per site, a Mott transition occurs and the nature of the possible Mott-insulating phases are determined.

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I. INTRODUCTION

Ultracold atomic physics has attracted a lot of interest in recent years with the opportunity to study strongly correlated effects, such as high-temperature superconductivity, in a new context

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[1]. In particular, loading cold atomic gases into an optical lattice allows for the realization of bosonic and fermionic lattice models and the experimental study of exotic quantum phases [2]. A prominent example is the toolbox to engineer the three-dimensional Bose-Hubbard model [3] and the observation of its Mott insulator – superfluid quantum phase transition with cold bosonic atoms in an optical lattice [4].

Ultracold atomic systems also offer an opportunity to investigate the effect of spin degeneracy since the atomic total angular momentum F, which includes both electron and nuclear spins, can be larger than 1/2 resulting in $2F+1$ hyperfine states. In magnetic traps, these $2F+1$ components are split, while in optical traps these (hyperfine) spin degrees of freedom are degenerate and novel interesting phases might emerge. For instance, Bose-Einstein condensates of bosonic atoms with a nonzero total spin F are expected to display rich interesting structures in spin space like in superfluid ^{3}He. In this respect, various exotic superfluid condensates (including nematic ones), Mott insulating phases, and non-trivial vortex structures have been predicted recently in spinor bosonic atoms with $F \geq 1$ [5, 6, 7, 8, 9, 10, 11]. These theoretical predictions might be checked in the context of Bose-Einstein condensates of sodium, rubidium atoms [12] and in spin-3 atom of ^{52}Cr [13]. The spin-degeneracy in fermionic atoms, like ^{6}Li, ^{40}K or ^{173}Yb, is also expected to give rise to some interesting superfluid phases [14, 15, 16, 17, 18]. In particular, a molecular superfluid (MS) phase might be stabilized in multicomponent attractive Fermi gas where more than two fermions form a bound state. Such a non-trivial superfluid behavior has already been found in different contexts. In nuclear physics, a four-particle condensate like the α particle is favored over deuteron condensation at low density [19] and it may have implications for light nuclei and asymmetric matter in nuclear stars [20, 21]. This quartet condensation can occur in the field of semiconductors with the formation of biexciton [22]. A quartetting phase, which stems from the pairing of Cooper pairs, also appears in a model of one-dimensional Josephson junctions [23]. A possible experimental observation of quartets might be found in superconducting quantum interference devices with (100)/(110) interfaces of two d-wave superconductors [24]. In particular, the $hc/4e$ periodicity of the critical current with applied magnetic flux has been interpreted as the formation of quartets with total charge $4e$ [25]. Finally, a bipairing mechanism has also been predicted in four-leg Hubbard ladders [26].

Recently, the emergence of quartet and triplet (three-body bound states) has been proposed to occur in the context of ultracold fermionic atoms [27, 28, 29, 30, 31, 32]. Much of these studies have been restricted to the special case of an SU(n) symmetry between the hyperfine states of a n-component Fermi gas. In this paper, we investigate the generic physical features of half-integer

spins $F = N - 1/2$ fermionic cold atoms with s-wave scattering interactions loaded into a one-dimensional optical lattice. The low-energy physical properties of $2F+1 = 2N$-component fermions with contact interactions are known to be described by a Hubbard-like Hamiltonian [5]:

$$\mathcal{H} = -t \sum_{i,\alpha} \left[c^\dagger_{\alpha,i} c_{\alpha,i+1} + \text{H.c.} \right] - \mu \sum_i n_i + \sum_{i,J} U_J \sum_{M=-J}^{J} P^\dagger_{JM,i} P_{JM,i}, \tag{1}$$

where $c^\dagger_{\alpha,i}$ ($\alpha = 1, ..., 2N$) is the fermion creation operator corresponding to the $2F+1 = 2N$ atomic states and $n_i = \sum_\alpha c^\dagger_{\alpha,i} c_{\alpha,i}$ is the density operator on site i. The pairing operators in Eq. (1) are defined through the Clebsch-Gordan coefficients for forming a total spin J from two spin-F fermions: $P^\dagger_{JM,i} = \sum_{\alpha\beta} \langle JM|F,F;\alpha\beta\rangle c^\dagger_{\alpha,i} c^\dagger_{\beta,i}$. The interactions are SU(2) spin-conserving and depend on U_J parameters corresponding to the total spin J which takes only even integers value due to Pauli's principle: $J = 0, 2, ..., 2N-2$. Even in this simple scheme the interaction pattern is still involved since there are N coupling constants in the general spin-F case. In this paper, we shall consider a two coupling-constant version of model (1) with $U_2 = ... = U_{2N-2} \neq U_0$ which incorporates the relevant physics of higher-spin degeneracy with respect to the formation of an exotic MS phase. To this end, it is enlightening to express this model in terms of the density n_i and the BCS singlet-pairing operator for spin-F which is defined by:

$$P^\dagger_{00,i} = \frac{1}{\sqrt{2N}} \sum_{\alpha,\beta} c^\dagger_{\alpha,i} \mathcal{J}_{\alpha\beta} c^\dagger_{\beta,i} = \frac{1}{\sqrt{2N}} \sum_\alpha (-1)^{2F+\alpha}\, c^\dagger_{\alpha,i} c^\dagger_{2N+1-\alpha,i}, \tag{2}$$

where the matrix $\mathcal{J}$ is the natural generalization of the familiar antisymmetric tensor $\epsilon = i\sigma_2$ to spin $F > 1/2$. Such a singlet-pairing operator has been extensively studied in the context of two-dimensional frustrated spin-1/2 quantum magnets [33]. Using the relation $\sum_{J,M} P^\dagger_{JM,i} P_{JM,i} = n_i^2$, model (1) with $U_2 = ... = U_{2N-2} \neq U_0$ reads then as follows:

$$\mathcal{H} = -t \sum_{i,\alpha} [c^\dagger_{\alpha,i} c_{\alpha,i+1} + \text{H.c.}] - \tilde{\mu} \sum_i n_i + \frac{U}{2} \sum_i n_i^2 + V \sum_i P^\dagger_{00,i} P_{00,i}, \tag{3}$$

with $U = 2U_2$, $V = U_0 - U_2$, and $\tilde{\mu} = \mu - U/2$.

When $V = 0$, i.e. $U_0 = U_2$, model (3) corresponds to the Hubbard model with an SU($2N$) spin symmetry: $c_{\alpha,i} \rightarrow \sum_\beta U_{\alpha\beta} c_{\beta,i}$, U being a SU($2N$) matrix. The Hamiltonian (3) for $V \neq 0$ still displays an extended symmetry. Indeed, since the BCS singlet-pairing operator (2) is invariant under the Sp($2N$) group which consists of unitary matrices U that satisfy $U^* \mathcal{J} U^\dagger = \mathcal{J}$, model (3) acquires an Sp($2N$) symmetry [34]. In the spin $F = 1/2$ case, model (3) reduces to the SU(2) Hubbard chain since SU(2) $\simeq$ Sp(2). In the $F = 3/2$ case, i.e. $N = 2$, there is no fine-tuning and models (1) and (3) have thus an exact Sp(4)$\simeq$ SO(5) symmetry [35]. For $F > 3/2$, the

Sp($2N$) symmetry is only present for the two coupling-constant model (3). The Sp($2N$) symmetry simplifies the problem but may appear rather artificial at this point. However, one reason to consider this model stems from the structure of the interaction involving the density n_i and the BCS singlet-pairing term (2) which are independent for $F \geq 3/2$ [1]. The competition between these two operators gives rise to interesting physics. Indeed, at $U = 0$ and large $V < 0$, a quasi-long-range BCS phase, which is a singlet under Sp($2N$), should emerge. In contrast, for $V = 0$ and $U < 0$, we expect the stabilization of a charge-density wave (CDW) or a MS instability made of $2N$ fermions $M_i = c^\dagger_{1,i} \ldots c^\dagger_{2N,i}$ which is an SU($2N$) singlet. We thus expect the existence of several distinct phases in the Sp($2N$) model (3). As it will be revealed in the following, the competition between these instabilities relies on the existence of a $\mathbb{Z}_N$ symmetry which is *also* a symmetry of the original SU(2) model (1). This $\mathbb{Z}_N$ symmetry can be defined by considering the coset between the center $\mathbb{Z}_{2N}$ of the SU($2N$) group [2] and the center $\mathbb{Z}_2$ of the Sp($2N$) or SU(2): $\mathbb{Z}_N = \mathbb{Z}_{2N}/\mathbb{Z}_2$ with

$$\mathbb{Z}_{2N} : c^\dagger_{\alpha,i} \;\rightarrow\; \mathrm{e}^{in\pi/N} c^\dagger_{\alpha,i} \;, n = 0, ..., 2N-1 \tag{4}$$

$$\mathbb{Z}_{N} : c^\dagger_{\alpha,i} \;\rightarrow\; \mathrm{e}^{in\pi/N} c^\dagger_{\alpha,i} \;, n = 0, ..., N-1, \tag{5}$$

the $\mathbb{Z}_2$ symmetry being $c_{\alpha,i} \rightarrow -c_{\alpha,i}$. The $\mathbb{Z}_N$ symmetry (5) provides an important physical ingredient which is not present in the $F = 1/2$ case and stems from the higher-spin degeneracy. The stabilization of a quasi-long-range BCS phase for $F > 1/2$ requires the spontaneous breaking of this $\mathbb{Z}_N$ symmetry since the singlet pairing $P^\dagger_{00,i}$ (2) is *not* invariant under this symmetry. In contrast, if $\mathbb{Z}_N$ is not broken, a MS phase, which is a singlet under the $\mathbb{Z}_N$ symmetry, might emerge. In the following, the delicate competition between these superfluid instabilities will be investigated with a special emphasis on this $\mathbb{Z}_N$ symmetry. In particular, in the low-energy approach, this discrete symmetry is described by the $\mathbb{Z}_N$ parafermionic conformal field theory (CFT) which captures the universal properties of two-dimensional $\mathbb{Z}_N$ generalized Ising models [36]. This approach will enable us to determine the main features of the zero temperature phase diagram of model (3). A brief summary of our main results have already been published as a letter [29].

The rest of the paper is organized as follows. The low-energy effective theory corresponding to model (3) is developed in section II. The nature of the different phases is then deduced from a renormalization group (RG) analysis. In section III, we present the phase diagram of the model

[1] For $F = 1/2$, one has indeed $P^\dagger_{00,i} P_{00,i} = n_i^2 - n_i$.

[2] We recall that the center of a group G is defined from the elements which commute with all elements of G. In particular, the Sp($2N$) or SU(2) groups share the same center which is the $\mathbb{Z}_2$ group.

(3) at zero temperature for incommensurate filling and we discuss the main physical properties of the phases as well as the nature of the quantum phase transitions. The Mott-insulating phases for the case of one atom per site are also investigated. Finally, we conclude in section IV and some technical details are presented in three appendices.

II. LOW-ENERGY APPROACH

In this section, we present the low-energy description of the spin-F Hubbard model with a Sp($2N$) symmetry (3) which will enable us to determine its phase diagram at zero temperature in the next section.

A. Continuum limit

Let us first discuss the continuum limit of the lattice model (3). Its low-energy effective field theory can be derived from the continuum description of the lattice fermionic operators $c_{\alpha,i}$ in terms of right and left-moving Dirac fermions (for a review, see for instance the books [37, 38, 39]):

$$c_{\alpha,i}/\sqrt{a_0} \rightarrow R_\alpha \mathrm{e}^{ik_F x} + L_\alpha \mathrm{e}^{-ik_F x}, \tag{6}$$

with $x = ia_0$ (a_0 being the lattice spacing) and k_F is the Fermi momentum. The left (right)-moving fermions are holomorphic (antiholomorphic) fields of the complex coordinate $z = v_F \tau + ix$ (τ being the imaginary time and v_F is the Fermi velocity): $R_\alpha(\bar{z}), L_\alpha(z)$. These fields obey the following operator product expansion (OPE):

$$\begin{aligned} R^\dagger_\alpha\left(\bar{z}\right) R_\beta\left(\bar{\omega}\right) &\sim \frac{\delta_{\alpha\beta}}{2\pi\left(\bar{z}-\bar{\omega}\right)} \\ L^\dagger_\alpha\left(z\right) L_\beta\left(\omega\right) &\sim \frac{\delta_{\alpha\beta}}{2\pi\left(z-\omega\right)}. \end{aligned} \tag{7}$$

In this continuum limit, the non-interacting part of the Hamiltonian (3) corresponds to the Hamiltonian density of $2N$ free relativistic massless fermions:

$$\mathcal{H}_0 = -iv_F \left(: R^\dagger_\alpha \partial_x R_\alpha : - : L^\dagger_\alpha \partial_x L_\alpha : \right), \tag{8}$$

where $v_F = 2ta_0 \sin(k_F a_0)$ and the normal ordering :: with respect to the Fermi sea is assumed as well as a summation over repeated indices. The continuous symmetry of the non-interacting part of the Hamiltonian (3) is enlarged, in the continuum limit, to $\mathrm{U}(2N)|_L \otimes \mathrm{U}(2N)|_R$ under independent unitary transformations on the Dirac fermions.

The next step of the approach is to use the non-Abelian bosonization [40, 41, 42] to investigate the effect of the Sp($2N$) symmetry of the lattice model. To this end, we introduce a SU($2N$)$_1$ Wess-Zumino-Novikov-Witten (WZNW) primary field $g_{\alpha\beta}$ to represent the spin degrees of freedom and an additional U(1) charge boson field Φ_c. The free-fermion theory (8) is then equivalent to the CFT with an U(1) $\times$ SU($2N$)$_1$ symmetry. The right and left-moving fermions can be written as a product of spin and charge operators:

$$\begin{aligned} R_\alpha &\sim \; : \exp\left(i\sqrt{2\pi/N}\;\Phi_{cR}\right) : g_{\alpha R} \\ L^\dagger_\alpha &\sim \; : \exp\left(i\sqrt{2\pi/N}\;\Phi_{cL}\right) : g_{\alpha L}, \end{aligned} \tag{9}$$

where g_R and g_L are respectively the right and left parts of the SU($2N$)$_1$ primary field g with scaling dimension $(2N-1)/2N$ which transforms in the fundamental representation of SU($2N$). In Eq. (9), $\Phi_{cR,L}$ denote the chiral parts of the charge boson field: $\Phi_c = \Phi_{cR} + \Phi_{cL}$. As is well known, the free-Hamiltonian (8) can be decomposed into charge and spin pieces:

$$\mathcal{H}_0 = \frac{v_F}{2}\left[(\partial_x\Phi_c)^2 + (\partial_x\Theta_c)^2\right] + \frac{2\pi v_F}{2N+1}\left[: I^A_R I^A_R : + : I^A_L I^A_L :\right], \tag{10}$$

where $\Theta_c = \Phi_{cL} - \Phi_{cR}$ is the dual charge boson field and $I^A_{R,L}(A = 1,\ldots,4N^2-1)$ are the currents which generate the SU($2N$)$_1$ conformal symmetry with central charge $c = 2N-1$. They express simply as bilinears of the Dirac fermions:

$$I^A_R =: R^\dagger_\alpha T^A_{\alpha\beta} R_\beta :, \;\; I^A_L =: L^\dagger_\alpha T^A_{\alpha\beta} L_\beta :, \tag{11}$$

T^A being the generators of SU($2N$) in the fundamental representation normalized such that: $\mathrm{Tr}(T^A T^B) = \delta^{AB}/2$ (see Appendix A). These currents satisfy the SU($2N$)$_1$ current algebra [38, 43]:

$$\begin{aligned} I^A_L(z)\, I^B_L(\omega) &\sim \frac{\delta^{AB}}{8\pi^2 (z-\omega)^2} + \frac{i f^{ABC}}{2\pi (z-\omega)} I^C_L(\omega) \\ I^A_R(\bar{z})\, I^B_R(\bar{\omega}) &\sim \frac{\delta^{AB}}{8\pi^2 (\bar{z}-\bar{\omega})^2} + \frac{i f^{ABC}}{2\pi (\bar{z}-\bar{\omega})} I^C_R(\bar{\omega}), \end{aligned} \tag{12}$$

where f^{ABC} are the structure constants of the SU($2N$) group.

In the presence of interactions, the spin-charge separation (10) still holds away from half-filling (i.e. N atoms per site). In that case, the low-energy Hamiltonian of model (3) separates into two commuting charge and spin pieces:

$$\mathcal{H} = \mathcal{H}_c + \mathcal{H}_s, \;\; [\mathcal{H}_c, \mathcal{H}_s] = 0. \tag{13}$$

Let us first consider the charge degrees of freedom which are described by the bosonic field Φ_c.

1. Charge degrees of freedom

Using the continuum description (6) and the decomposition (9), we find that the charge excitations of model (3) are captured by the following Hamiltonian:

$$\mathcal{H}_c = \frac{v_F}{2}\left[(\partial_x\Phi_c)^2 + (\partial_x\Theta_c)^2\right] + a_0\frac{2V+UN\,(2N-1)}{2N\pi}\,(\partial_x\Phi_c)^2\,. \qquad (14)$$

It can be recast into the form of the Tomonaga-Luttinger free Hamiltonian:

$$\mathcal{H}_c = \frac{v_c}{2}\left[\frac{1}{K_c}\,(\partial_x\Phi_c)^2 + K_c\,(\partial_x\Theta_c)^2\right], \qquad (15)$$

where the charge velocity v_c and the Luttinger parameter K_c read as follows:

$$\begin{aligned} v_c &= v_F\left[1 + a_0(2V+UN(2N-1))/(N\pi v_F)\right]^{1/2} \\ K_c &= \left[1 + a_0(2V+UN(2N-1))/(N\pi v_F)\right]^{-1/2}. \end{aligned} \qquad (16)$$

The conserved quantities in this U(1) charge sector are the total particle number $\mathcal{N}$ and current $\mathcal{I}$:

$$\begin{aligned} \mathcal{N} &= \int dx\ :R^\dagger_\alpha R_\alpha + L^\dagger_\alpha L_\alpha: = \sqrt{2N/\pi}\int dx\ \partial_x\Phi_c \\ \mathcal{I} &= \int dx\ :L^\dagger_\alpha L_\alpha - R^\dagger_\alpha R_\alpha := \sqrt{2N/\pi}\int dx\ \partial_x\Theta_c. \end{aligned} \qquad (17)$$

For generic fillings, no umklapp terms appear and the charge degrees of freedom display metallic (gapless) properties in the Luttinger liquid universality class [38, 39]. However, as it will be discussed in section III, the existence of an umklapp process, for a commensurate filling of one atom per site, is responsible for the formation of a charge gap and the emergence of different Mott insulating phases.

2. Spin-degrees of freedom

All non-trivial physics corresponding to the spin degeneracy is encoded in the spin part $\mathcal{H}_s$ of the Hamiltonian (13). Its continuum expression can be obtained by decomposing the $\mathrm{SU}(2N)_1$ currents of Eq. (11) into $\parallel$ and $\perp$ parts $I^A_{R,L} = (I^a_\parallel, I^i_\perp)_{R,L}$ with respect to the Sp($2N$) symmetry of the lattice model (3). The currents $I^a_{\parallel R(L)}, a = 1,..,N(2N+1)$ generate the $\mathrm{Sp}(2N)_1$ CFT symmetry with central charge $c = N(2N+1)/(N+2)$. They can be simply expressed in terms of the chiral Dirac fermions:

$$I^a_{\parallel R} =: R^\dagger_\alpha T^a_{\alpha\beta} R_\beta :,\ \ I^a_{\parallel L} =: L^\dagger_\alpha T^a_{\alpha\beta} L_\beta :, \qquad (18)$$

where T^a are the generators of Sp($2N$) in the fundamental representation and normalized such that: $\mathrm{Tr}(T^a T^b) = \delta^{ab}/2$ (see Appendix A). These currents verify the Sp$(2N)_1$ Kac-Moody algebra given by Eqs. (12) with f^{abc} the Sp($2N$) structure constants. The remaining SU$(2N)_1$ currents $I^i_{\perp R,L}$ $(i = 1,..,2N^2 - N - 1)$ are written as:

$$I^i_{\perp R} =: R^\dagger_\alpha T^i_{\alpha\beta} R_\beta :, \;\; I^i_{\perp L} =: L^\dagger_\alpha T^i_{\alpha\beta} L_\beta : . \tag{19}$$

With these definitions, we can now derive the continuum description of the spin degrees of freedom using Eq. (6) and Eqs. (A2, A8) of Appendix A. After some cumbersome calculations, we find that the low-energy Hamiltonian in the spin sector, $\mathcal{H}_s$, can be expressed in terms of the currents only, and displays a marginal current-current interaction:

$$\mathcal{H}_s = \frac{2\pi v_{s\parallel}}{2N+1}\left[: I^a_{\parallel R} I^a_{\parallel R} : + : I^a_{\parallel L} I^a_{\parallel L} :\right] + \frac{2\pi v_{s\perp}}{2N+1}\left[: I^i_{\perp R} I^i_{\perp R} : + : I^i_{\perp L} I^i_{\perp L} :\right] + g_\parallel I^a_{\parallel R} I^a_{\parallel L} + g_\perp I^i_{\perp R} I^i_{\perp L}, \tag{20}$$

with $g_\parallel = -2a_0(2V + NU)/N$, $g_\perp = 2a_0(2V - NU)/N$ and $v_{s\parallel,\perp}$ are the spin velocities: $v_{s\parallel} = v_F - Ua_0/2\pi - Va_0/\pi$, $v_{s\perp} = v_F - Ua_0/2\pi + Va_0(N+1)/N\pi$. In the simplest $N = 1$ case, we have Sp$(2)_1 \sim$ SU$(2)_1$ and the Hamiltonian (20) with $g_\perp = 0$ describes the spin-sector of the continuum limit of the spin-1/2 Hubbard chain which can be found for instance in the book [38] written with the same notations used in this work. In the $N = 2$, i.e. the $F = 3/2$ case, one can express model (20) in a more transparent basis. Indeed, there is a simple free-field representation of the unperturbed SU$(4)_1 \sim$ SO$(6)_1$ CFT in terms of six real (Majorana) fermions which has been used in the context of the two-leg spin-1/2 ladder with four-spin exchange interactions [44]. Introducing six real fermions $\xi^0_{R,L}$ and $\xi^i_{R,L}, i = 1,...,5$ to describe respectively the $\mathbb{Z}_2$, i.e. Ising, and SO$(5)_1 \simeq$ Sp$(4)_1$ CFTs, the currents of Eq. (20) can be written locally in terms of these fermions:

$$\begin{aligned} I^i_{\perp R,L} &\sim -\frac{i}{\sqrt{2}} \xi^0_{R,L} \xi^i_{R,L} \\ I^a_{\parallel R,L} &\sim -\frac{i}{\sqrt{2}} \xi^i_{R,L} \xi^j_{R,L}, \end{aligned} \tag{21}$$

where $a = 1, \ldots, 10 \equiv (i,j)$ where $1 \leq i < j \leq 5$. The Majorana fermions are normalized as the Dirac fermions of Eq. (7) to reproduce faithfully the Kac-Moody algebra (12). The current-current model of Eq. (20) can then be expressed in terms of these real fermions:

$$\mathcal{H}_s = -\frac{iv}{2}\left[: \xi^i_R \partial_x \xi^i_R : - : \xi^i_L \partial_x \xi^i_L :\right] - \frac{iv_0}{2}\left[: \xi^0_R \partial_x \xi^0_R : - : \xi^0_L \partial_x \xi^0_L :\right] + \lambda_\parallel (\xi^i_R \xi^i_L)^2 + \lambda_\perp \xi^0_R \xi^0_L \xi^i_R \xi^i_L, \tag{22}$$

with $\lambda_\parallel = -a_0(U + V)$, $\lambda_\perp = a_0(V - U)$, and the spin velocities: $v = v_F - a_0(U + V)/2\pi$, $v_0 = v_F - a_0(U - 3V)/2\pi$. In absence of the spin-velocity anisotropy, model (22) turns out to be exactly solvable and has also been studied in the context of a SO(5) symmetric two-leg ladder [45].

B. RG analysis

In the general N case, model (20) is not integrable and the main effect of the current-current interaction can be elucidated by means of a RG analysis. The one-loop RG equations of model (20) are given by (see Appendix B):

$$\begin{aligned} \dot{g}_{\|} &= \frac{N+1}{4\pi}\, g_{\|}^2 + \frac{N-1}{4\pi}\, g_{\perp}^2 \\ \dot{g}_{\perp} &= \frac{N}{2\pi}\, g_{\perp} g_{\|}, \end{aligned} \tag{23}$$

where $\dot{g}_{\perp,\|} = \partial g_{\perp,\|}/\partial t$, t being the RG time. In Eq. (23), we have neglected the spin-velocity anisotropy $v_s = v_{s\|} \simeq v_{s\perp}$ and have absorbed v_s in a redefinition of the coupling constants: $g_\alpha \rightarrow g_\alpha / v_s$. We have also obtained the two-loop RG equations of model (20) and the results are presented in Appendix B. The one-loop RG equations (23) can be solved and in particular the RG invariant flow K which parametrizes the RG lines reads as follows:

$$K = g_{\perp}^{-(N+1)/N} \left(g_{\|}^2 - g_{\perp}^2 \right). \tag{24}$$

The RG flow emerging from Eqs. (23) is rich and consists of three different phases (see Fig. 1).

In region I, where both U and V are positive, all couplings go to zero in the infrared (IR) limit and the interaction is marginal irrelevant. The symmetry of the IR fixed point is SU$(2N)_1$ (up to a velocity anisotropy) leading to $2N-1$ gapless spin excitations. The current-current interaction of Eq. (20) leads to logarithmic corrections to physical quantities [46, 47]. The low-energy properties of this phase are very similar to that of the repulsive SU($2N$) Hubbard chain which have been studied in Refs. [48, 49]. In contrast, a spin gap is opened by the interaction in the two remaining phases. In phase II, defined by $U < 0$ and $V > NU/2$, the RG flow in the far IR limit is attracted along a special symmetric ray $g_{\|} = g_{\perp} = g^* > 0$ where the interacting part of the Hamiltonian (20) can be rewritten in a manifest SU($2N$) invariant form:

$$\mathcal{H}^*_{\rm s,int} = g^* \left(I^a_{\|R} I^a_{\|L} + I^i_{\perp R} I^i_{\perp L} \right) = g^* I^A_R I^A_L. \tag{25}$$

This IR Hamiltonian thus takes the form of the SU($2N$) Gross-Neveu (GN) model [50] which is an integrable massive field theory [51]. The development of the strong-coupling regime in the SU($2N$) GN model leads to the generation of a spin gap. The low-energy properties of the spin sector of phase II can be extracted from the integrability of the SU($2N$) GN model (25). Its low-energy spectrum consists into $2N-1$ branches with masses: $m_r = m \sin(\pi r/2N)$ that transform in the SU($2N$) representation with Young tableau with one column and r boxes ($r = 1, \ldots, 2N-1$) [51].

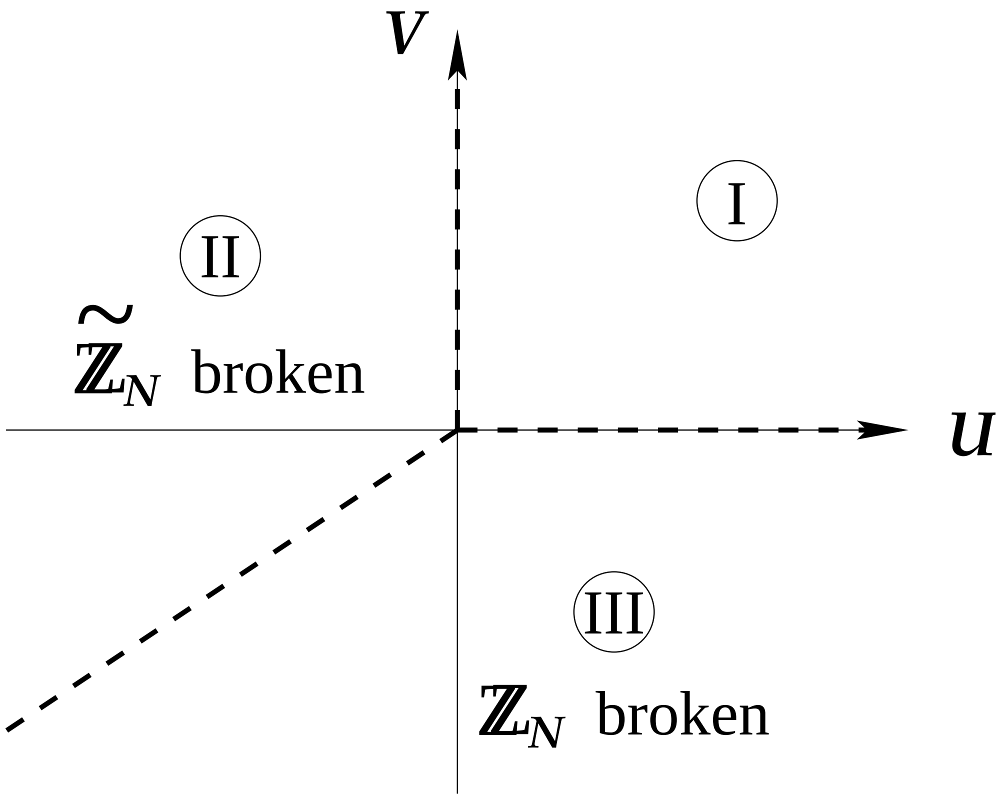

FIG. 1: Phase diagram of model (20) obtained from the RG approach. Phase I is gapless while phases II and III have gapped spin excitations. The dash lines stand for the phase transition lines predicted from the RG calculation.

The corresponding eigenstates are labelled by quantum numbers associated with the conserved quantities of the SU($2N$) low-energy symmetry (the Cartan basis): $Q^a_{\parallel} = \int dx \; (I^a_{\parallel L} + I^a_{\parallel R})$, $a = (1, ..., N)$ and $Q^i_{\perp} = \int dx \; (I^i_{\perp L} + I^i_{\perp R})$, $i = (1, ..., N-1)$. Due to the Sp($2N$) symmetry of model (3), the $Q^a_{\parallel}$ numbers are conserved whereas the $Q^i_{\perp}$ charges are only good quantum numbers at low energy. This is an example of a dynamical symmetry enlargement which corresponds to a situation where a Hamiltonian is attracted under a RG flow to a manifold possessing a higher symmetry than indicated by the original microscopic theory. This phenomenon occurs in a large variety of models with marginal interactions in the scaling limit [52] as, for instance, in the half-filled two-leg Hubbard model [53] and in the SU(4) Hubbard model at half filling [54], where a SO(8) symmetry emerges at low energy.

In the second spin-gapped phase (III) of Fig. 1, defined by $V < 0$ and $V < NU/2$, the RG flow is now attracted along the asymptote: $g_{\parallel} = -g_{\perp} = g^* > 0$. In that case, the interacting part of

the IR Hamiltonian becomes

$$\mathcal{H}^{*}_{\mathrm{s,int}} = g^{*} \left(I^{a}_{\parallel R} I^{a}_{\parallel L} - I^{i}_{\perp R} I^{i}_{\perp L} \right), \tag{26}$$

which can be recast as a SU($2N$) GN model (25) by means of a *duality* transformation $\mathcal{D}$ on the fermions: $\mathcal{D} R(L) \mathcal{D}^{-1} = \tilde{R}(\tilde{L})$ with

$$\tilde{R}_{\alpha} = \mathcal{J}_{\alpha\beta} R^{\dagger}_{\beta}, \quad \tilde{L}_{\alpha} = L_{\alpha}. \tag{27}$$

Using Eqs. (18, 19), we observe that this transformation acts on the currents as: $\tilde{I}^{a}_{\parallel R(L)} = I^{a}_{\parallel R(L)}$ and $\tilde{I}^{i}_{\perp R(L)} = -(+) I^{i}_{\perp R(L)}$ so that $\mathcal{D}$ indeed maps (26) onto (25). Besides the opening of a spectral gap, we thus find that phase III possesses a hidden symmetry at low energy i.e. a $\widetilde{SU}(2N)$ symmetry generated by the dual currents $(\tilde{I}^{a}_{\parallel R(L)}, \tilde{I}^{i}_{\perp R(L)})$. The spin spectrum in phase III can be obtained from the duality symmetry $\mathcal{D}$ and consists into the $2N-1$ branches m_r which transform in the representations of the dual group $\widetilde{SU}(2N)$. The dual quantum numbers are now given by: $\tilde{Q}^{a}_{\parallel} = Q^{a}_{\parallel}$ and $\tilde{Q}^{i}_{\perp} = \int dx \; (\tilde{I}^{i}_{\perp L} + \tilde{I}^{i}_{\perp R}) = \int dx \; (I^{i}_{\perp L} - I^{i}_{\perp R}) = \mathcal{I}^{i}_{\perp}$. We thus observe that the low-lying excitations in phase III carry quantized spin *currents* in the "$\perp$" direction. In this sense, the phase III might be viewed as a partially spin-superfluid phase. In summary, the existence of these two distinct spin-gapped phases is a non-trivial consequence of higher-spin degeneracy and does not occur in the $F = 1/2$ case.

C. Conformal embedding

The crucial point of a non-perturbative analysis is often the identification of a good basis that describe low-lying excitations of the phase. Much insight on this problem can be gained from the symmetries of the model and the use of the non-Abelian bosonization approach for 1D systems. Such an approach has been extremely powerful in the past as in quantum impurity problems [55] and in spin chains [38, 42]. So far, we have used an U($2N$)$_1$ = U(1) × SU($2N$)$_1$ CFT approach to determine the low-energy properties of model (3). However, this description is not adequate to give a full understanding of the two spin-gapped phases found in the RG approach. In particular, the physical origin of the formation of the spin gaps is not clear at this point. What is the nature of the discrete symmetry which is spontaneously broken in phases II and III? A second weak point of the previous analysis is the determination of the quantum phase transition between the two spin-gapped phases which is rather unclear within the preceding description. At least, this transition should occur in the manifold which is invariant under the duality $\mathcal{D}$ symmetry (27) i.e. the self-dual

manifold defined by: $g_\perp = 0$. The low-energy Hamiltonian which describes the phase transition is thus (neglecting the spin-velocity anisotropy):

$$\mathcal{H}_s^{\rm SD} = \frac{2\pi v_s}{2N+1}\left[: I^a_{\parallel R} I^a_{\parallel R} : + : I^a_{\parallel L} I^a_{\parallel L} : + : I^i_{\perp R} I^i_{\perp R} : + : I^i_{\perp L} I^i_{\perp L} :\right] + g_\parallel^{\rm SD} I^a_{\parallel R} I^a_{\parallel L}, \tag{28}$$

with $g_\parallel^{\rm SD} > 0$ so that the marginally relevant current-current interaction opens a mass gap in the Sp($2N$) sector. However, one cannot conclude on the occurrence of the first-order phase transition since the non-interacting Hamiltonian of Eq. (28) contains more degrees of freedom than the Sp($2N$) one. They remain massless and control the quantum phase transition. The nature of these decoupled degrees of freedom is not clear at this point.

Prompted by all these questions, it is important to fully exploit the existence of the Sp($2N$) symmetry of the lattice model (3) and to consider the following conformal embedding: U$(2N)_1$ $\rightarrow$ U(1) $\times$ Sp$(2N)_1$ $\times$ [SU$(2N)_1$/Sp$(2N)_1$]. The coset SU$(2N)_1$/Sp$(2N)_1$ CFT has central charge $c = 2N - 1 - N(2N+1)/(N+2) = 2(N-1)/(N+2)$ which is that of the $\mathbb{Z}_N$ parafermionic CFT [36]. In fact, as shown by Altschuler [56], it turns out that the SU(2N)$_1$/Sp(2N)$_1$ CFT is indeed equivalent to the $\mathbb{Z}_N$ parafermionic CFT which describes self-dual critical points of two-dimensional $\mathbb{Z}_N$ Ising models (see Appendix C). The conformal embedding U$(2N)_1$ $\rightarrow$ U(1) $\times$ Sp$(2N)_1$ $\times$ $\mathbb{Z}_N$ provides us with a *non-perturbative* basis to express any physical operator in terms of its charge and spin degrees of freedom which are described respectively by the U(1) and Sp$(2N)_1$ $\times$ $\mathbb{Z}_N$ CFTs. Since Sp(2) $\sim$ SU(2), this basis for $N = 1$ accounts for the well-known spin-charge separation which is the hallmark of 1D spin-1/2 electronic systems [38, 39]. In this respect, in the general half-odd integer spin case, the $\mathbb{Z}_N$ symmetry plays its trick and provides a new important ingredient not present for $F = 1/2$. In the low-energy approach, the spin degrees of freedom corresponding to this symmetry are captured by an effective 2D $\mathbb{Z}_N$ Ising model. As in the $N = 2$ case, these $\mathbb{Z}_N$ Ising models exhibit two gapped phases described by order and disorder parameters σ_k and μ_k ($k = 1, .., N-1$) which are dual to each other by means of the Kramers-Wannier (KW) duality symmetry. This duality transformation maps the $\mathbb{Z}_N$ symmetry, spontaneously broken in the low-temperature phase ($\langle \sigma_k \rangle \neq 0$ and $\langle \mu_k \rangle = 0$), onto a $\tilde{\mathbb{Z}}_N$ symmetry which is broken in the high-temperature phase where $\langle \mu_k \rangle \neq 0$ and $\langle \sigma_k \rangle = 0$. At the critical point, the theory is self-dual with a $\mathbb{Z}_N \times \tilde{\mathbb{Z}}_N$ symmetry and its universal properties are captured by the $\mathbb{Z}_N$ parafermionic CFT with σ_k, μ_k becoming conformal fields with scaling dimension $d_k = k(N-k)/N(N+2)$ [36]. This $\mathbb{Z}_N$ CFT is generated by right and left parafermionic currents $\Psi_{kR,L}$ ($\Psi^\dagger_{kR,L} = \Psi_{N-kR,L}$, $k = 1, \ldots, N-1$) with scaling dimension $\Delta_k = k(N-k)/N$ which are the generalization of the Majorana fermions of the $\mathbb{Z}_2$ Ising model. Under the $\mathbb{Z}_N \times \tilde{\mathbb{Z}}_N$ symmetry, Ψ_{kL} (respectively Ψ_{kR})

carries a (k,k) (respectively $(k,-k)$) charge which means:

$$\begin{aligned}\Psi_{kL,R} &\rightarrow e^{i2\pi mk/N}\Psi_{kL,R} \text{ under } \mathbb{Z}_N \\ \Psi_{kL,R} &\rightarrow e^{\pm i2\pi mk/N}\Psi_{kL,R} \text{ under } \tilde{\mathbb{Z}}_N,\end{aligned} \tag{29}$$

with $m = 0,\ldots,N-1$. As it is discussed in Appendix C, there is a faithful representation of the first parafermionic currents $\Psi_{1L,R}$ in terms of the Dirac fermions and the charge bosonic field $\Phi_{cR,L}$:

$$\begin{aligned}L^\dagger_\alpha \mathcal{J}_{\alpha\beta} L^\dagger_\beta &\simeq \frac{\sqrt{N}}{\pi} : \exp\left(i\sqrt{8\pi/N}\;\Phi_{cL}\right) : \Psi_{1L} \\ R^\dagger_\alpha \mathcal{J}_{\alpha\beta} R^\dagger_\beta &\simeq \frac{\sqrt{N}}{\pi} : \exp\left(-i\sqrt{8\pi/N}\;\Phi_{cR}\right) : \Psi_{1R}.\end{aligned} \tag{30}$$

We deduce, from this identification, that the duality symmetry (27) of the RG analysis together with the Gaussian duality ($\Phi_{cR,L} \rightarrow \mp\Phi_{cR,L}$) give $\Psi_{1L} \rightarrow \Psi_{1L}$ and $\Psi_{1R} \rightarrow -\Psi^\dagger_{1R}$ which is nothing but the KW duality transformation on the first parafermionic current (see Eq. (C2) of Appendix C). It is thus tempting to interpret the formation of the spin-gapped phases of Fig. 1 as the result of the spontaneous breaking of the $\mathbb{Z}_N$ and $\tilde{\mathbb{Z}}_N$ discrete symmetries. In this respect, from the correspondence (30) and Eq. (29), we observe that under the $\mathbb{Z}_N$ symmetry, the left and right-moving fermions transform as:

$$L_\alpha \rightarrow e^{-i\pi m/N} L_\alpha, \;\; R_\alpha \rightarrow e^{-i\pi m/N} R_\alpha, \tag{31}$$

while under $\tilde{\mathbb{Z}}_N$ we have:

$$L_\alpha \rightarrow e^{-i\pi m/N} L_\alpha, \;\; R_\alpha \rightarrow e^{i\pi m/N} R_\alpha. \tag{32}$$

Using the continuum representation of the fermions (6), we find that the lattice $\mathbb{Z}_N$ symmetry (5) corresponds to the $\mathbb{Z}_N$ symmetry of an effective 2D $\mathbb{Z}_N$ Ising model. In contrast, the $\tilde{\mathbb{Z}}_N$ symmetry has no simple local lattice representation.

We need now to fully identify phases II and III of Fig. 1 with the high and low-temperature phases of the $\mathbb{Z}_N$ Ising model. To this end, we use the fact that the SU$(2N)_1$ WZNW primary field g of Eq. (9) can be expressed in terms of the Sp$(2N)_1 \times \mathbb{Z}_N$ basis using the conformal embedding (see Eq. (C9) of Appendix C):

$$: \exp\left(-i\sqrt{2\pi/N}\Phi_c\right) : L^\dagger_\alpha R_\alpha \sim \text{Tr } g \sim \mu_1 \text{Tr } \phi^{(1)}, \tag{33}$$

where $\phi^{(1)}$ is the Sp$(2N)_1$ primary field with scaling dimension $\Delta = (2N+1)/2(N+2)$. We deduce, from the identification (33), that Tr g has the same behavior as μ_1 under the $\mathbb{Z}_N \times \tilde{\mathbb{Z}}_N$ symmetry

(Eq. (C1) of Appendix C):

$$\begin{aligned} \mathrm{Tr}\ g &\rightarrow \mathrm{Tr}\ g \ \text{ under } \ \mathbb{Z}_N \\ \mathrm{Tr}\ g &\rightarrow e^{i2\pi m/N} \mathrm{Tr}\ g \ \text{ under } \ \tilde{\mathbb{Z}}_N. \end{aligned} \tag{34}$$

The next step of the approach is to note that the interacting part of the Hamiltonian (25), which controls the strong coupling behavior of the RG flow in phase II, can be expressed in terms of Tr g: $\mathcal{H}^*_{s,\mathrm{int}} \sim |\mathrm{Tr}\ g|^2$. The ground state of this phase displays long-range order associated with the order parameter Tr g: $\langle \mathrm{Tr}\ g \rangle \neq 0$. According to Eq. (34), we then deduce that the $\tilde{\mathbb{Z}}_N$ symmetry is spontaneously broken while the $\mathbb{Z}_N$ symmetry remains unbroken in phase II. The $\mathbb{Z}_N$ Ising model thus belongs to its high-temperature phase and a spectral gap is formed. Using the KW duality symmetry or the transformation (27) on the Dirac fermions, one can conclude that phase III corresponds to the low-temperature phase of the $\mathbb{Z}_N$ Ising model where the $\mathbb{Z}_N$ symmetry is spontaneously broken.

In summary, the existence of the two spin-gapped phases of Fig. 1 is a non-trivial consequence of higher-spin degeneracy and does not occur in the $F = 1/2$ case. The emergence of the spin-gap stems from the spontaneous breakdown of the $\mathbb{Z}_N$ or $\tilde{\mathbb{Z}}_N$ discrete symmetries. As we shall see now, these symmetries are central to the striking physical properties displayed by these phases.

III. PHASE DIAGRAM

In this section, we discuss the phase diagram at zero temperature of the lattice model (3) for incommensurate filling and for a commensurate filling of one atom per site. The nature of the quantum phase transitions will also be investigated. Let us start with the incommensurate filling case.

A. Phase diagram for incommensurate filling

We shall now determine the nature of the dominant electronic instabilities of the different phases of Fig. 1 for incommensurate filling. The RG analysis of the preceding section reveals the existence of three different phases.

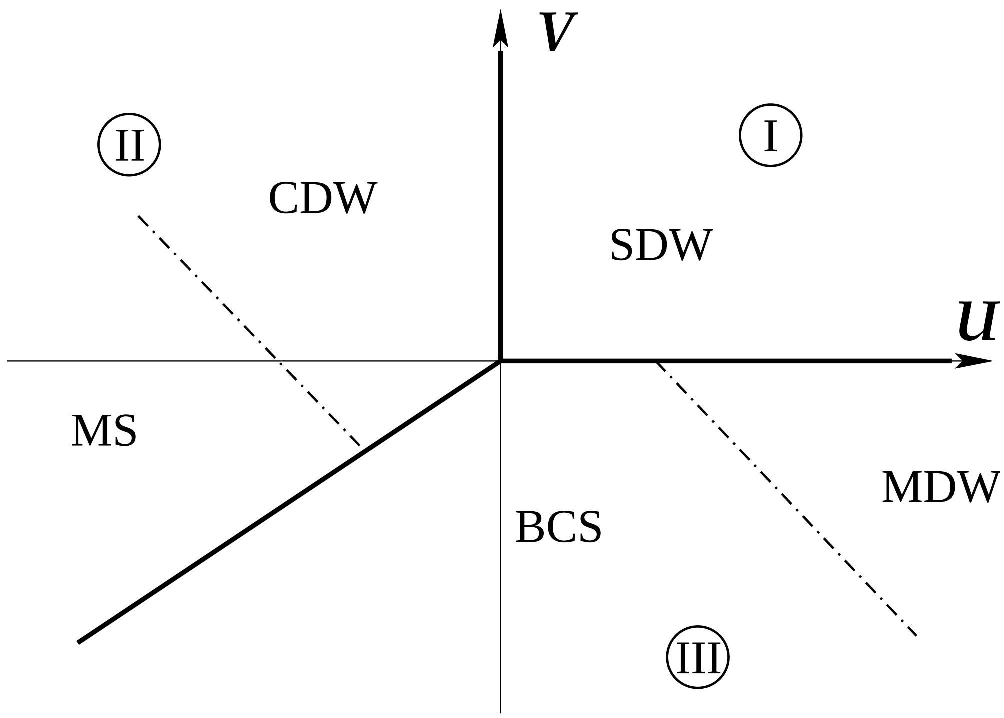

FIG. 2: Phase diagram of model (3) for incommensurate filling.

1. Critical phase

In phase I of Fig. 2, the interaction in the spin sector (Eq. (20)) is marginally irrelevant when U and V are both positive so that the phase displays an extended quantum critical behavior. Up to a spin-velocity anisotropy, the low-energy properties of this phase are very similar to that of the repulsive SU($2N$) Hubbard chain with one gapless charge mode and $2N-1$ gapless spin excitations [48, 49]. All correlation functions in this phase display a power-law behavior. The leading instabilities are those which have the slowest decaying correlations at long distance. For phase I, the dominant instabilities are the $2k_F$ CDW ρ_{2k_F} and generalized $2k_F$ spin-density wave (SDW) $\mathcal{S}^A_{2k_F}$ order parameters which read as follows in terms of the Dirac fermions:

$$\rho_{2k_F} = L^{\dagger}_{\alpha} R_{\alpha}, \;\; \mathcal{S}^A_{2k_F} = L^{\dagger}_{\alpha} T^A_{\alpha\beta} R_{\beta}. \tag{35}$$

Using the representation (C9), we obtain the leading asymptotics of the $2k_F$ CDW correlation function:

$$\langle \rho^{\dagger}_{2k_F}(x,\tau)\, \rho_{2k_F}(0,0) \rangle \sim \left(x^2 + v_c^2\tau^2\right)^{-K_c/2N} \left(x^2 + v^2\tau^2\right)^{-(2N+1)/2(N+2)} \left(x^2 + v_0^2\tau^2\right)^{-(N-1)/N(N+2)}, \tag{36}$$

where v and v_0 are respectively the spin-velocity in the Sp($2N$) and $\mathbb{Z}_N$ sectors. We have obtained a similar estimate for the correlations which involve the $2k_F$ SDW operator. At this point, these two correlation functions have the same power-law decay. The logarithmic corrections will lift this degeneracy and we expect that the SDW operator will be the dominant instability as in the $N = 1$ case [38, 39].

2. Confined phase

Let us now consider the first spin-gapped phase, i.e. phase II of Fig. 1, which occurs when $U < 0$ and $V > NU/2$ in the weak-coupling limit. In contrast to phase I, it has only one gapless mode which stems from the criticality of the charge degrees of freedom. The existence of a spin gap leads us to expect the emergence of a quasi-long-range BCS phase with the pairing of fermions, i.e a Luther-Emery liquid phase, as in the $F = 1/2$ case [38, 39]. However, this is not the case for $F \geq 3/2$ due to the existence of the $\mathbb{Z}_N$ symmetry (5) which remains unbroken in phase II. Indeed, it costs a finite energy gap to excite states that break this symmetry and the dominant instabilities must thus be neutral under $\mathbb{Z}_N$. In particular, there is no dominant BCS instability in phase II since, as already stated in the introduction, the lattice singlet-pairing operator $P^{\dagger}_{00,i}$ (2) is not invariant under the $\mathbb{Z}_N$ symmetry (5). Another way to see this is to use the low-energy description of the BCS operator obtained in Appendix C (Eq. (C10)):

$$P^{\dagger}_{00} \sim L^{\dagger}_{\alpha} \mathcal{J}_{\alpha\beta} R^{\dagger}_{\beta} \sim : \exp\left(i\sqrt{2\pi/N}\Theta_c\right) : \sigma_1 \mathrm{Tr}\ \phi^{(1)}, \tag{37}$$

where Tr $\phi^{(1)}$ takes a non-zero expectation value in phase II. Since the $\mathbb{Z}_N$ symmetry of the underlying Ising model is not broken, the $\mathbb{Z}_N$ Ising spin variable σ_1 has zero expectation value and short range correlations. Consequently, the equal-time correlation function of the BCS instability (37) has a short-range behavior: $\langle P^{\dagger}_{00}(x)\, P_{00}(0)\rangle \sim e^{-x/\xi}$. The BCS singlet pairing is completely suppressed in this phase.

In contrast, the dominant instabilities in phase II are expected to be:

$$\rho_{2k_F} = L^{\dagger}_{\alpha} R_{\alpha}, \;\; M_0 = \epsilon^{\alpha_1 \ldots \beta_N} R^{\dagger}_{\alpha_1} \ldots R^{\dagger}_{\alpha_N} L^{\dagger}_{\beta_1} \ldots L^{\dagger}_{\beta_N}, \tag{38}$$

which are respectively the $2k_F$ CDW and the uniform component of the lattice SU($2N$)-singlet superconducting instability made of $2N$ fermions, i.e. the MS instability, $M_i = c^{\dagger}_{1,i}..c^{\dagger}_{2N,i}$. Both order parameters (38) are neutral under the $\mathbb{Z}_N$ symmetry (31) and we notice that M_0 is also invariant under the $\tilde{\mathbb{Z}}_N$ symmetry (32). The long-distance behavior of the correlation functions of

these operators can be determined using the identifications (C9) and (C13, C14) of Appendix C:

$$\rho_{2k_F} \sim \, :\exp\left(i\sqrt{2\pi/N}\Phi_c\right): \mu_1 \mathrm{Tr}\, \phi^{(1)} \tag{39}$$

$$M_0 \sim \, :\exp\left(i\sqrt{2\pi N}\Theta_c\right): \sum_{p=0}^{N/2} a_p \, \epsilon_{N/2-p} \, \mathrm{Tr}\, \phi^{(2p)}, \tag{40}$$

where we have assumed, for the sake of simplicity, that N is even for the representation of the MS instability. The fields in the $\mathrm{Sp}(2N) \times \mathbb{Z}_N$ sector, which occur in these expressions, have non-zero expectation values in phase II so that:

$$\rho_{2k_F} \sim \, :\exp\left(i\sqrt{2\pi/N}\Phi_c\right): \tag{41}$$

$$M_0 \sim \, :\exp\left(i\sqrt{2\pi N}\Theta_c\right):. \tag{42}$$

We then deduce that these orders have power-law decaying equal-time correlation functions:

$$\langle \rho^{\dagger}_{2k_F}(x)\, \rho_{2k_F}(0)\rangle \sim x^{-K_c/N}, \tag{43}$$

$$\langle M^{\dagger}_0(x)\, M_0(0)\rangle \sim x^{-N/K_c}. \tag{44}$$

We thus see that CDW and MS instabilities compete and the key point of the analysis is the one which dominates. At issue is the value of the Luttinger parameter K_c. For $K_c < N$, the leading instability is ρ_{2k_F} which gives rise to a CDW phase whereas for $K_c > N$ a MS phase is stabilized with a lattice order parameter M_i. Such a large value of K_c is not guaranteed for fermionic models with only short-range interactions as model (3). The full non-perturbative behavior of the Luttinger parameter as a function of the interactions U, V and density n is beyond scope of the low-energy approach. Its value should be determined numerically by means of Quantum Monte Carlo (QMC) technique or the density matrix renormalization group (DMRG) calculations. Such analysis has been performed recently for the spin-3/2 case, i.e. $N = 2$, and reveals that the MS phase (a quartetting phase in that case) exists for a wide range of attractive contact interactions at sufficiently small density [30]. In the general N case, a strong-coupling investigation at small density shows that an upper bound for K_c is $K_{c\max} = N$ [30]. From the perturbative estimate (16), we see that $K_c > 1$ for attractive interactions and increases with $|U|$ and $|V|$ so that we expect the emergence of a MS phase in the general N case for sufficiently strong attractive interactions and small density. Such a phase is displayed on Fig. 2 and the doted line marks the crossover line between the CDW and MS phases.

The MS phase with the formation of bound-states made of $2N$ fermions is remarkable and is a consequance of the higher-spin degeneracy when $F \geq 3/2$. The onset of this phase stems from the

existence of the $\mathbb{Z}_N$ symmetry which remains unbroken in phase II. This discrete symmetry leads to the suppression of the charge $2e$ BCS superconducting instability and confines the electronic charge to multiple of $2Ne$ i.e. the leading superfluid instability is a composite object made of $2N$ fermions. In this respect, the MS phase might be viewed as a kind of nematic Luther-Emery liquid since SU(2) invariant spin-1/2 electronic systems with a spin gap and gapless charge degrees of freedom, i.e. Luther-Emery liquids, exhibit a pairing phase [57].

3. *Deconfined phase*

The properties of phase III are obtained from those of phase II with help of the duality symmetry $\mathcal{D}$ (27) on the fermions or ($\Phi_c \leftrightarrow \Theta_c, \mathbb{Z}_N \leftrightarrow \tilde{\mathbb{Z}}_N$) in terms of the charge and spin degrees of freedom. Phase III has only one gapless charge mode and the $\mathbb{Z}_N$ symmetry, in the spin sector, is now spontaneously broken. The effective two-dimensional $\mathbb{Z}_N$ Ising model belongs to its low-temperature phase where the $\tilde{\mathbb{Z}}_N$ remains unbroken. We find now the emergence of a quasi-long-range BCS pairing phase which is described by the order parameter (37). Since σ_1 and $\mathrm{Tr}\phi^{(1)}$ acquire now a non-zero expectation value in phase III, we have the following leading behavior at low-energy:

$$P_{00}^{\dagger} \sim : \exp\left(i\sqrt{2\pi/N}\;\Theta_c\right) :, \tag{45}$$

which leads to the power-law decaying equal-time correlation function:

$$\langle P_{00}^{\dagger}\left(x\right) P_{00}\left(0\right)\rangle \sim x^{-1/NK_c}. \tag{46}$$

The $\mathbb{Z}_N$ symmetry, being spontaneously broken in phase III, does not confine anymore the electronic charge and thus accounts for the emergence of a BCS superfluid phase. In the spin-3/2 case, this phase has been found numerically by means of QMC and DMRG calculations [58]. The MS instability, being neutral under $\mathbb{Z}_N$, still has the power-law decay (44) in phase III and is always subdominant with respect to the BCS operator (45). In contrast, the latter singlet pairing instability is now competing with the operator $\bar{\rho}_{2Nk_F}^{(2N)}$ which is obtained from the MS instability (38) by the duality transformation (27):

$$\bar{\rho}_{2Nk_F}^{(2N)} = \epsilon^{\alpha_1 ... \alpha_N \beta_1 ... \beta_N} \mathcal{J}_{\alpha_1\gamma_1} ... \mathcal{J}_{\alpha_N\gamma_N}\; R_{\gamma_1} ... R_{\gamma_N} L_{\beta_1}^{\dagger} ... L_{\beta_N}^{\dagger}, \tag{47}$$

which corresponds to a molecular density-wave (MDW) phase, with wave vector $2Nk_F$, made of $2N$ fermions. Its low-energy description can be derived from Eq. (40) by means of the Gaussian

duality ($\Phi_c \leftrightarrow \Theta_c$) and the KW transformation:

$$\bar{\rho}^{(2N)}_{2Nk_F} \sim : \exp\left(i\sqrt{2\pi N}\Phi_c\right) : \sum_{p=0}^{N/2} a_p \; (-1)^{N/2-p} \, \epsilon_{N/2-p} \; \mathrm{Tr} \; \phi^{(2p)}, \tag{48}$$

where we have used the KW transformation of the thermal operators of the $\mathbb{Z}_N$ CFT: $\epsilon_j \rightarrow (-1)^j \epsilon_j$. The Sp($2N$) $\times$ $\mathbb{Z}_N$ fields in Eq. (48) have non-zero expectation values in phase III so that the leading asymptotics of the MDW correlation reads as follows:

$$\langle \bar{\rho}^{(2N)\,\dagger}_{2Nk_F} (x) \, \bar{\rho}^{(2N)}_{2Nk_F} (0) \rangle \sim x^{-NK_c}. \tag{49}$$

Using Eq. (46), we conclude that the BCS instability is the dominant one of phase III when $K_c > 1/N$ whereas the MDW phase emerges for $K_c < 1/N$. The crossover line between these two phases is denoted by dotted lines in Fig. 2.

4. *Quantum phase transition*

As it has been discussed in section II, the nature of the quantum phase transition between the two spin-gapped phases II and III can be determined through the duality symmetry $\mathcal{D}$ (27). On the self-dual line $g_\perp = 0$, i.e. $2V = NU$, the low-energy properties of the transition are captured by the field theory (28). Using the conformal embedding approach, we observe that there is a separation of the Sp($2N$) and $\mathbb{Z}_N$ degrees of freedom in Eq. (28). Though the Sp($2N$) sector remains gapfull when $U < 0$, the effective $\mathbb{Z}_N$ Ising model is at its self-dual critical point and governs the phase transition. The $\mathbb{Z}_N$ quantum criticality might be revealed explicitly by considering the following ratio $\mathcal{R}_N$ at equal times:

$$\mathcal{R}_N \, (x) = \left(\langle P^\dagger_{00} \, (x) \, P_{00} \, (0) \rangle \right)^{N^2} / \langle M^\dagger_0 \, (x) \, M_0 \, (0) \rangle. \tag{50}$$

In phase II, this ratio admits an exponential decay since the singlet-pairing instability is short range. Using Eqs. (44, 46), we find that $\mathcal{R}_N \, (x) \sim$ cte in phase III. At the $\mathbb{Z}_N$ quantum critical point, $\mathcal{R}_N \, (x)$ has a power-law behavior. Indeed, at this point, the BCS instability (37) simplifies as follows:

$$P^\dagger_{00} \, (x) \sim : \exp\left(i\sqrt{2\pi/N}\Theta_c\right) : \sigma_1, \tag{51}$$

where we have averaged the Sp($2N$) degrees of freedom. Similarly, using the identification (40), we obtain that the MS instability $M_0 \, (x)$ is still given by Eq. (42). Therefore, we find that, at the $\mathbb{Z}_N$ critical point, $\mathcal{R}_N \, (x)$ has a power-law decay with a *universal* exponent:

$$\mathcal{R}_N \, (x) \sim x^{-2N(N-1)/(N+2)}, \tag{52}$$

i.e. an exponent 1 and 12/5 respectively for the Ising ($N = 2$) and three-state Potts ($N = 3$) cases. The long-distance behavior of the function $\mathcal{R}_2(x)$ close to the transition has been determined recently numerically by means of large scale DMRG calculations and the emergence of the $\mathbb{Z}_2$ quantum criticality has been revealed [58]. However, the phase transition may be non-universal for larger N since we have neglected irrelevant perturbations in the weak-coupling limit $|U, V| \ll t$ that could become relevant at the quantum phase transition. Some insights might be gained from the symmetries of the problem. The relevant operators should belong to the $\mathbb{Z}_N$ sector, be neutral under the $\mathbb{Z}_N \times \tilde{\mathbb{Z}}_N$ symmetry and invariant under the KW transformation. The natural candidates are the thermal operators ϵ_j ($j = 1, \ldots, N/2$) of the $\mathbb{Z}_N$ CFT with scaling dimension $2j(j+1)/(N+2)$ which transforms as $\epsilon_j \to (-1)^j \epsilon_j$ under the KW duality symmetry [36]. We thus deduce that, in the $N = 2, 3$ cases, the quantum phase transition between phases II and III is universal and belongs to the Ising and three-state Potts universality classes respectively. For $N \geq 4$, a strongly relevant perturbation ϵ_2 with scaling dimension $12/(N+2)$ is generated in the $\mathbb{Z}_N$ sector. The resulting field theory which captures the quantum phase transition for $F \geq 7/2$ becomes:

$$\mathcal{S}_{\text{eff}} = \mathcal{S}_{\mathbb{Z}_N} + \lambda \int d^2x \; \epsilon_2(x) \,, \tag{53}$$

where $\mathcal{S}_{\mathbb{Z}_N}$ stands for the action of the $\mathbb{Z}_N$ CFT. Model (53) turns out to be an integrable deformation of the $\mathbb{Z}_N$ CFT [59]. The nature of the phase transition depends on the sign of the coupling constant λ [59]. For $\lambda < 0$, the field theories (53) are massive and the phase transition is of first-order type. For $\lambda > 0$ it is known that model (53) has a massless RG flow onto a Kosterlitz-Thouless (KT) U(1) gapless phase with central charge $c = 1$.

Finally, we end this subsection by briefly considering the other phase transitions that occur in Fig. 2. The quantum phase transition between phases I and III is of KT type and its exact position is $V = 0$ and $U > 0$, as it will be seen in section III B 4, which corresponds to the repulsive SU($2N$) Hubbard model. The transition between phases I and II is also expected to be of the KT type but its actual position is not clear. Within the weak-coupling approach of section II, its location is $U = 0$ and $V > 0$ but higher-order corrections might bend this line.

B. Commensurate filling: one-atom per site

We now discuss the nature of the phase diagram of model (3) for a commensurate $1/2N$ filling ($k_F = \pi/(2Na_0)$) i.e. one atom per site. In this case, there is still a spin-charge separation (13)

and the main modification occurs in the charge sector with the existence of an umklapp process which appears at higher order of the perturbation theory, and that transfers $2N$ fermions from one Fermi point to the other, i.e. $L_1^\dagger R_1 L_2^\dagger R_2 \ldots L_{2N}^\dagger R_{2N} + \mathrm{H.c.}$. The low-energy Hamiltonian, which describes the charge degrees of freedom, becomes now a sine-Gordon model:

$$\mathcal{H}_c = \frac{v_c}{2}\left[\frac{1}{K_c}\left(\partial_x \Phi_c\right)^2 + K_c\left(\partial_x \Theta_c\right)^2\right] - g_c \cos\left(\sqrt{8\pi N}\, \Phi_c\right). \tag{54}$$

Such umklapp operator can also be determined from a symmetry analysis. Indeed, for the commensurate filling $k_F = \pi/(2Na_0)$, using the representation (6), the lattice one-step translation symmetry $\mathcal{T}_{a_0}$ reads as follows on the Dirac fermions:

$$L_\alpha \rightarrow e^{-i\pi/2N} L_\alpha, \;\; R_\alpha \rightarrow e^{i\pi/2N} R_\alpha, \tag{55}$$

so that we deduce $\Phi_c \rightarrow \Phi_c + \sqrt{\pi/2N}$ under $\mathcal{T}_{a_0}$ from Eq. (9). The cosine term of Eq. (54) is thus the one with the smallest scaling dimension and compatible with the translation symmetry. Since its scaling dimension is $\Delta_u = 2NK_c$, we deduce that a charge gap is opened when $K_c < 1/N$. A Mott transition occurs between a Luttinger phase and an insulating phase (see Fig. 3). In the latter phase, the charge bosonic field Φ_c is pinned in one of the minima of the sine-Gordon model (54):

$$\begin{aligned} \langle \Phi_c \rangle &= \sqrt{\frac{\pi}{2N}}\, m, \;\; g_c > 0 \\ \langle \Phi_c \rangle &= \sqrt{\frac{\pi}{2N}}\left(m + \frac{1}{2}\right), \;\; g_c < 0, \end{aligned} \tag{56}$$

m being integer. Three different Mott-insulating phases can then be defined depending on the spin degrees of freedom and the status of the $\mathbb{Z}_N$ discrete symmetry (see Fig. 3).

1. *Mott I phase*

In the first Mott phase, the spin degrees of freedom remain gapless. This phase is qualitatively similar to the insulating phase of the repulsive SU($2N$) Hubbard chain at $1/(2N)$ filling with $2N-1$ gapless bosonic spin modes [48, 49]. In particular, it includes (when taking the limit of large repulsive U) the SU(2N) generalization of the Heisenberg model:

$$\mathcal{H} = J \sum_i S_i^A S_{i+1}^A, \tag{57}$$

where S_i^A are the SU($2N$) spin operators on site i: $S_i^A = c_{\alpha,i}^\dagger T_{\alpha\beta}^A c_{\beta,i}$. This model is integrable by means of the Bethe ansatz approach [60] and its critical properties are captured by the SU$(2N)_1$

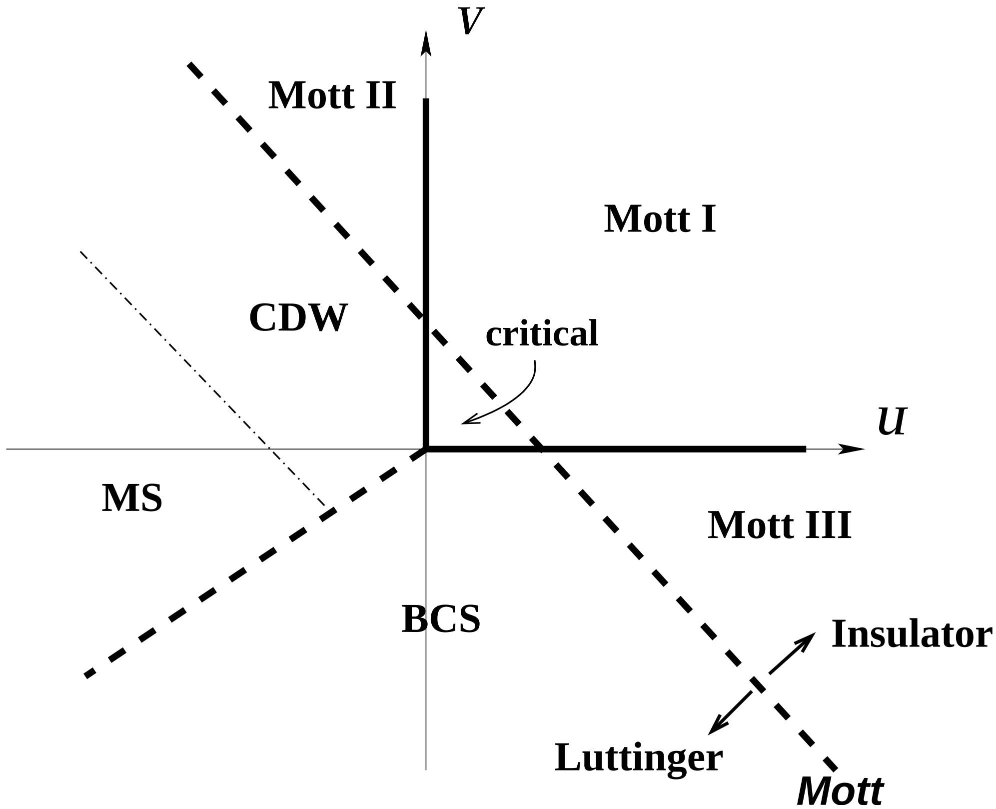

FIG. 3: Phase diagram of model (3) for a filling of one atom per site.

WZNW model [48]. The leading asymptotics of the spin-spin correlation functions in this phase can be easily obtained from Eq. (36) by gapping the charge degrees of freedom:

$$\langle S^A(x,\tau)\, S^B(0,0)\rangle \sim \frac{\delta^{AB}\cos(2k_F x)}{\left(x^2+v^2\tau^2\right)^{(2N+1)/2(N+2)}\left(x^2+v_0^2\tau^2\right)^{(N-1)/N(N+2)}}. \tag{58}$$

2. Mott II phase

The remaining Mott insulating phases of Fig. 3 are fully gapped. In region II, we have seen in section III that the $\mathbb{Z}_N$ (respectively $\tilde{\mathbb{Z}}_N$) symmetry in the spin sector is unbroken (respectively spontaneously broken). We expect that the lattice translation symmetry $\mathcal{T}_{a_0}$ is spontaneously broken in the Mott II phase. This can be explicitly shown by considering the lattice $2N$-merization operator which is defined by:

$$\Delta_{2k_F} = e^{-i\pi j/N} S_j^A S_{j+1}^A. \tag{59}$$

In the continuum limit, this order parameter expresses directly in terms of the SU$(2N)_1$ primary field: $\Delta_{2k_F} \sim \mathrm{Tr} g \sim \mu_1 \mathrm{Tr}\phi^{(1)}$. Since the $\tilde{\mathbb{Z}}_N$ symmetry is broken, i.e. $\langle \mu_1 \rangle \neq 0$, Tr$g$ acquires a

non-zero expectation value in this phase: $\langle \Delta_{2k_F} \rangle \neq 0$. We deduce, from the definition (59), that the translation symmetry is spontaneously broken and the Mott II phase displays a $2N$-merization with a $2N$ ground-state degeneracy. On top of this order, we also expect the formation of a long-range ordering of a $2\mathrm{k}_F$ CDW with order parameter $\rho = \cos(\pi j/N)\ c^{\dagger}_{\alpha,j} c_{\alpha,j}$. Using Eqs. (6,41), we get the following identification in phase II:

$$\rho \simeq L^{\dagger}_{\alpha} R_{\alpha} + R^{\dagger}_{\alpha} L_{\alpha} \sim : \cos\left(\sqrt{2\pi/N}\ \Phi_c\right) :, \tag{60}$$

which has a non-zero expectation value due to the pinning of the charge bosonic field (56). The $2\mathrm{k}_F$ CDW coexists with the $2N$-merization operator (59) in the Mott II phase.

3. Mott III phase

The Mott transition line $K_c = 1/N$ coincides with the cross-over line between BCS and MDW phases in Fig. 2. For a filling of one atom per site, the BCS phase is still present while the MDW phase becomes a fully gapped Mott insulating phase. In this Mott phase, the $\mathbb{Z}_N$ symmetry is now spontaneously broken while the $\tilde{\mathbb{Z}}_N$ one remains unbroken. In the continuum limit, an order parameter of this phase is provided by the $2Nk_F$ MDW operator (47) or the $2Nk_F$ CDW: $\rho_{2Nk_F} = \epsilon^{\alpha_1 \ldots \alpha_N \beta_1 \ldots \beta_N} L^{\dagger}_{\alpha_1} \ldots L^{\dagger}_{\alpha_N} R_{\beta_1} \ldots R_{\beta_N}$. In particular, using the character decomposition (C12) of Appendix C, we get:

$$\rho_{2Nk_F} \sim: \exp\left(i\sqrt{2\pi N}\Phi_c\right) : \sum_{p=0}^{N/2} a_p\ \epsilon_{N/2-p}\ \mathrm{Tr}\ \phi^{(2p)}. \tag{61}$$

After averaging out the $\mathrm{Sp}(2N)_1 \times \mathbb{Z}_N$ fields, $2Nk_F$ MDW (48) and $2Nk_F$ CDW order parameters have the same leading behavior which is given by:

$$\rho_{2Nk_F} \sim: \exp\left(i\sqrt{2\pi N}\ \Phi_c\right) : . \tag{62}$$

Using the position of the pinning of the charge bosonic field (56), we deduce that $\langle \mathrm{Re}\ \rho_{2Nk_F} \rangle \neq 0$ (respectively $\langle \mathrm{Im}\ \rho_{2Nk_F} \rangle \neq 0$) when $g_c > 0$ (respectively $g_c < 0$). The electronic properties of this phase thus depends on the sign of the coupling constant g_c of the umklapp operator in Eq. (54). This sign might be fixed using higher-order perturbation theory as in Ref. [61] or numerically by looking at the different lattice order parameters of the problem. In this respect, for $g_c > 0$, we have the long-range ordering of a $q = 2Nk_F = \pi/a_0$ CDW: $\langle \rho_\pi \rangle = \langle \sum_{j,\alpha} (-1)^j c^{\dagger}_{j,\alpha} c_{j,\alpha} \rangle \neq 0$ while for $g_c < 0$ a spin-Peierls (bond) ordering is formed: $\langle \mathcal{O}_{\mathrm{SP}} \rangle = \langle \sum_{j,\alpha} (-1)^j c^{\dagger}_{j+1,\alpha} c_{j,\alpha} + \mathrm{H.c.} \rangle \neq 0$. In any case, the Mott III phase is two-fold degenerate for all N and the translation symmetry is broken.

This Mott phase is quite unusual since there is no one-particle charge density long-range fluctuation due to the spontaneously breaking of the $\mathbb{Z}_N$ symmetry. In the $N = 2$ case, at quarter-filling, it has been found numerically by means of the DMRG approach that a Mott transition occurs with the formation of a $4k_F = \pi/a_0$ bond ordering insulating phase [58]. We thus expect, at least for $N = 2$, that the coupling constant of the umklapp process is negative in phase III: $g_c < 0$. Finally, we observe that the Mott III phase is the only fully gapped phase directly connected to the BCS phase (see Fig. 3).

4. Strong coupling expansion

At large positive U, charge degrees of freedom are completely frozen by a large Mott gap, and it is possible to perform a strong coupling expansion that leads to an effective model for the low-energy spin degrees of freedom. Indeed, when $t/U = 0$ in Eq. (3), the ground states with energy 0 (the chemical potential is set to $\mu = U/2$), which consist of states with all sites filled by exactly one particle ('singly-occupied states'), are well separated in energy from all other states by a large gap U. One should note that the gap to those states that are singly occupied on all sites except on two sites, one being empty and the other one being doubly occupied by a SU(2) singlet $P^\dagger_{00}\left|0\right\rangle$, is $U + V$. Therefore the following picture holds only when $t \ll U, U + V$ to avoid formation of BCS pairs that occurs at large negative V. Nevertheless, we shall see that the BCS phase is captured by this approach, since it extends up to $V = 0^-$.

The huge degeneracy of the ground-state manifold consisting of singly occupied states is lifted by the hopping term $\mathcal{H}_0 = -t\sum_{i,\alpha} c^\dagger_{\alpha,i} c_{\alpha,i} + \text{H.c.}$ as soon as t/U departs from 0. To second order (the first non-trivial order) in perturbation theory, the effective Hamiltonian acting on singly occupied states reads:

$$\mathcal{H}_{\text{eff}} = \mathcal{P}\mathcal{H}_0(1-\mathcal{P})\frac{1}{\mathcal{H}_{U,V}}(1-\mathcal{P})\mathcal{H}_0 P, \tag{63}$$

where $\mathcal{P}$ is the projector onto singly occupied states and $\mathcal{H}_{U,V} = \mathcal{H} - \mathcal{H}_0$ is the quartic part of the Hamiltonian (3). The effective Hamiltonian can then be expressed in terms of SU($2N$) spin operators defined on each site, $S^A_i = c^\dagger_{\alpha,i} T^A_{\alpha\beta} c_{\beta,i}$, out of which we can single out the S^a_i ($a = 1, \ldots, N(2N + 1)$) that constitute a basis of the Sp($2N$) spin operators. One finds:

$$\mathcal{H}_{eff} = G_0 \sum_i \sum_{A=1}^{4N^2-1} S^A_i S^A_{i+1} + G_1 \sum_i \sum_{a=1}^{N(2N+1)} S^a_i S^a_{i+1}, \tag{64}$$

with the couplings $G_0 = \frac{t^2}{N}\left(\frac{2N-1}{U} + \frac{1}{U+V}\right)$ and $G_1 = \frac{t^2}{N}\left(\frac{1}{U} - \frac{1}{U+V}\right)$.

When $V = 0$, $G_1 = 0$ and the model reduces to the SU($2N$) antiferromagnet in the fundamental representation. As recalled in section III B 1, this integrable model is described at low energy by the SU($2N$)$_1$ WZNW model perturbed by a marginally irrelevant current-current interaction. At small V/U, i.e. in the vicinity of the SU($2N$) invariant point, we can thus obtain a continuum limit of Hamiltonian (64) in the form of a perturbed SU($2N$)$_1$ WZNW model. The calculation relies on the continuous representation of the spin operators, $S_j^A \simeq S^A(x = ja_0) = I_L^A + I_R^A + \sum_{m=1}^{2N-1} \alpha_m e^{2imk_F x} \mathcal{S}_{2mk_F}^A$, where the numbers α_m are non-universal amplitudes.

This procedure yields a low-energy Hamiltonian density exactly of the form (20), with couplings $g_\parallel(U,V)$ and $g_\perp(U,V)$. The actual value of $g_\parallel$ and $g_\perp$ are not known (they depend on non-universal amplitudes), but it is sufficient to know that $g_\parallel(U,0) = g_\perp(U,0) < 0$ at the SU($2N$) invariant point, and that $g_\parallel - g_\perp \propto G_1 \propto V$ at small V/U. The RG analysis of section II B thus leads to the conclusion that the spin sector undergoes a transition at $V = 0$, between a quantum critical phase ($V > 0$), that coincides with the Mott phase I, and a spin gapped phase ($V < 0$) with unbroken $\tilde{\mathbb{Z}}_N$ symmetry, which is Mott phase III. The strong coupling expansion thus establishes that the exact position of the transition line between these two Mott phases is $U > 0, V = 0$.

IV. CONCLUDING REMARKS

In summary, we have studied the zero-temperature phase diagram of one-dimensional spin-$F = N - 1/2$ fermionic cold atoms with contact interactions. In the low-energy limit, a CFT approach has been developed to deduce the main physical properties of the model by exploiting the presence of an extended Sp($2N$) symmetry. In the $F = 3/2$ case, this Sp(4) $\sim$ SO(5) symmetry is exact on the lattice while for larger F its existence stems from a fine-tuning of the coupling constants i.e. the scattering lengths of the atoms.

The phase diagram of the model is very rich for incommensurate filling with the competition between instabilities of very different nature. In particular, two different superfluid phases are found for attractive interactions: a Sp($2N$)-singlet BCS pairing phase and a MS phase. The latter phase is an SU($2N$) singlet which is formed from bound states of $2N$ fermions. For instance, for $N = 2$, i.e. $F = 3/2$, it is the analogous of the α particle in nuclear physics. At the heart of this competition is a $\mathbb{Z}_N$ discrete symmetry which is the coset between the center of the SU($2N$) group and the center of the Sp($2N$) one. Since Sp($2N$) and SU(2) share the same center group, this $\mathbb{Z}_N$ symmetry is not an artifact of the extended Sp($2N$) symmetry for $F \geq 7/2$ and is present for general SU(2)-invariant multicomponent Fermi gas. The $\mathbb{Z}_N$ symmetry plays a crucial role and

provides a new important ingredient which is not present for $F = 1/2$.

In the low-energy approach, this discrete symmetry corresponds to the symmetry of the low-temperature phase of an effective two-dimensional $\mathbb{Z}_N$ generalized Ising model. If the $\mathbb{Z}_N$ symmetry is not spontaneously broken, we find the suppression of the singlet-pairing BCS phase with the confinement of the electronic charge e to multiple of $2Ne$. In this case, the nature of the dominant instability of this phase depends on the non-universal Luttinger parameter K_c. If $K_c < N$, the leading instability is a $2k_F$ CDW while a quasi-long-range MS phase emerges for $K_c > N$ and should be the generic phase for attractive interactions at sufficiently small density. Due to its confinement properties and the existence of a spin gap, this MS phase can also be viewed as nematic Luther-Emery liquid. When the $\mathbb{Z}_N$ symmetry is spontaneously broken, a quasi-long-range BCS phase is stabilized when $K_c > 1/N$ whereas a MDW phase appears for $K_c < 1/N$. The quantum phase transition between the confined and BCS phases is described by the $\mathbb{Z}_N$ parafermionic CFT perturbed by the second thermal operator. For $F = 3/2$ and $5/2$, the transition is universal and belongs respectively to the Ising and three-state Potts universality classes. For higher spins $F \geq 7/2$, the transition is non-universal and is either of first-order or in the U(1) KT universality class.

For commensurate filling of one atom per site, in sharp contrast to the $F = 1/2$ case, a Mott transition is predicted to occur for $F \geq 3/2$ as the result of the spin degeneracy. Three different Mott-insulating phases are expected which depend on the status of the $\mathbb{Z}_N$ symmetry. A Mott phase with $2N - 1$ gapless spin modes is first found which is qualitatively similar to the insulating phase of the SU($2N$) Hubbard chain. The two others have a spin gap and display spin-Peierls (bond) ordering with a $2N$ or a two-fold ground-state degeneracy.

Regarding perspectives, it will be interesting to investigate numerically the phase diagram for $F = 5/2$ spins by means of large-scale DMRG or QMC calculations [62]. In particular, the three-state Potts universal behavior of the quantum phase transition might be confirmed numerically as well as the study of the Mott transition and the different Mott-insulating phases for one atom per site. From the theoretical point of view, another important question is the phase diagram for incommensurate filling of the SU(2) model (1) for $N \geq 3$ without any Sp($2N$) fine-tuning. Finally, we hope that the exotic MS phase for general $F = N - 1/2$ spins, discussed in this paper, will be observed in future experiments in ultracold spinor fermionic atoms.

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APPENDIX A: GROUP THEORY CONVENTIONS

The aim of this Appendix is to define the algebraic conventions used in the low-energy approach as well as the RG calculations which are presented in Appendix B.

Let us start with the SU($2N$) group. The generators of the Lie algebra of SU($2N$) in the fundamental representation ($2N \times 2N$ matrices) are denoted by T^A ($A = 1, \ldots, 4N^2 - 1$) with the normalization: $\mathrm{Tr}(T^A T^B) = \delta^{AB}/2$. They satisfy the commutation relation:

$$\left[T^A, T^B\right] = i f^{ABC} T^C, \tag{A1}$$

f^{ABC} being the structure constants of SU($2N$). An useful group identity for the derivation of the continuum limit of the lattice model (3) is:

$$\sum_A T^A_{\alpha\beta} T^A_{\gamma\rho} = \frac{1}{2}\left(\delta_{\alpha\rho}\delta_{\beta\gamma} - \frac{1}{2N}\,\delta_{\alpha\beta}\delta_{\gamma\rho}\right). \tag{A2}$$

The SU($2N$) generators can be divided in generators of the subalgebra Sp($2N$), T^a ($a = 1, \ldots, N(2N+1)$), and generators of the complement of Sp($2N$) in SU($2N$): T^i ($i = 1, \ldots, 2N^2 - N - 1$). They are normalized as: $\mathrm{Tr}(T^a T^b) = \delta^{ab}/2$, $\mathrm{Tr}(T^i T^j) = \delta^{ij}/2$. The various structure constants are then given by:

$$\left[T^a, T^b\right] = i f^{abc} T^c, \left[T^i, T^j\right] = i f^{ija} T^a, \left[T^a, T^i\right] = i f^{aij} T^j, \tag{A3}$$

which defines a $\mathbb{Z}_2$ graduation of the SU($2N$) Lie algebra. These coefficients can be determined by expressing the generators T^a, T^i in terms of a direct product between SU(N) and SU(2) generators. The SU(N) generators admit a simple $N \times N$ matrix representation which falls into three categories:

- Symmetric part:

$$\left(M^{(1)}_{ij}\right)_{\alpha\beta} = \frac{1}{2}(\delta_{i\alpha}\delta_{j\beta} + \delta_{i\beta}\delta_{j\alpha}) \qquad (1 \le i < j \le N) \tag{A4}$$

- Antisymmetric part:

$$\left(M^{(2)}_{ij}\right)_{\alpha\beta} = -\frac{i}{2}(\delta_{i\alpha}\delta_{j\beta} - \delta_{i\beta}\delta_{j\alpha}) \qquad (1 \le i < j \le N) \tag{A5}$$

- Cartan generators:

$$\left(M_m^{\mathrm{D}}\right)_{\alpha\beta} = \frac{1}{\sqrt{2m(m+1)}} \left(\sum_{k=1}^{m} \delta_{\alpha k}\delta_{\beta k} - m\,\delta_{\alpha,m+1}\delta_{\beta,m+1} \right), (m = 1, \ldots, N-1). \tag{A6}$$

The Sp($2N$) generators T^a ($a = 1, \ldots, N(2N+1)$) are then given by the following set:

$$T^a = \left\{ \frac{1}{\sqrt{2}}\, M_{ij}^{(1)} \otimes \vec{\sigma},\ \frac{1}{\sqrt{2}}\, M_{ij}^{(2)} \otimes I_2,\ \frac{1}{\sqrt{2}}\, M_m^{\mathrm{D}} \otimes \vec{\sigma},\ \frac{1}{2\sqrt{N}} I_N \otimes \vec{\sigma} \right\}, \tag{A7}$$

$\vec{\sigma}$ being the Pauli matrices. The metric $\mathcal{J}_{\alpha\beta}$, which defines the Sp($2N$) group, takes a simple form in this scheme: $\mathcal{J} = I_N \otimes (-i\sigma_2)$, and is thus the generalization of the antisymmetric tensor $\epsilon_{\alpha\beta}$ for higher spin $F > 1/2$. Using the representation (A7), we also obtain the following identity which is useful for the derivation of the low-energy approach of section II:

$$\sum_a T^a_{\alpha\beta} T^a_{\gamma\rho} = \frac{1}{4}\left(\delta_{\alpha\rho}\delta_{\beta\gamma} - \mathcal{J}_{\alpha\gamma}\mathcal{J}_{\beta\rho}\right). \tag{A8}$$

Finally, the remaining generators T^i ($i = 1, \ldots, 2N^2 - N - 1$) read as follows:

$$T^i = \left\{ \frac{1}{\sqrt{2}}\, M_{ij}^{(2)} \otimes \vec{\sigma},\ \frac{1}{\sqrt{2}}\, M_{ij}^{(1)} \otimes I_2,\ \frac{1}{\sqrt{2}}\, M_m^{\mathrm{D}} \otimes I_2 \right\}. \tag{A9}$$

APPENDIX B: TWO-LOOP RG RESULTS

In this Appendix, we present the two-loop RG equations of the current-current model (20):

$$\begin{aligned} \dot{g}_{\parallel} &= \left(1 - \frac{g_{\parallel}}{4\pi}\right)\left[(N+1)\,\frac{g_{\parallel}^2}{4\pi} + (N-1)\,\frac{g_{\perp}^2}{4\pi}\right] \\ \dot{g}_{\perp} &= N\frac{g_{\perp}g_{\parallel}}{2\pi} - \frac{N g_{\perp}}{(4\pi)^2}\left(g_{\parallel}^2 + g_{\perp}^2\right), \end{aligned} \tag{B1}$$

where $\dot{g}_{\perp,\parallel} = \partial g_{\perp,\parallel}/\partial t$, t being the RG time and we have neglected the spin-velocity anisotropy. To establish this result, we use the notations of Ref. [63] (we modify normalizations for our purpose) and define the operators:

$$\mathcal{O}_{\parallel}(z,\bar{z}) = I^a_{\parallel R}(\bar{z}) I^a_{\parallel L}(z), \quad \mathcal{O}_{\perp}(z,\bar{z}) = I^i_{\perp R}(\bar{z}) I^i_{\perp L}(z), \tag{B2}$$

in terms of which the interacting part of the action reads

$$S_I = \int d^2x\ g_\alpha \mathcal{O}_\alpha. \tag{B3}$$

Structure constants $C, D, \tilde{C}$ are defined for the algebra of the local operators $\mathcal{O}_\alpha$ and associated pseudo stress-energy tensors $\mathcal{T}_\parallel(z) =: I^a_{\parallel L} I^a_{\parallel L} : (z)$ and $\mathcal{T}_\perp(z) =: I^i_{\perp L} I^i_{\perp L} : (z)$, according to the OPE's:

$$\mathcal{O}_\alpha(z,\bar{z})\mathcal{O}_\beta(0,0) \sim \frac{1}{8\pi^2\,|z|^2}\, C^\gamma_{\alpha\beta}\mathcal{O}_\gamma(0,0), \tag{B4}$$

$$\mathcal{T}_\alpha(z)\mathcal{O}_\beta(0,0) \sim \frac{1}{8\pi^2\, z^2}\, \left(2kD^\gamma_{\alpha\beta} + \tilde{C}^\gamma_{\alpha\beta}\right)\mathcal{O}_\gamma(0,0). \tag{B5}$$

Here, to disentangle the structure constants D and $\tilde{C}$, we need to vary fictitiously the level k of the Kac-Moody algebras ($k = 1$ in our case), thus considering the more general problem of perturbing a $\mathrm{SU}(2N)_k$ model by marginal current-current interactions built on the currents of the $\mathrm{Sp}(2N)_k$ that are embedded in it. We still write $I^A_{L,R}$ for the $\mathrm{SU}(2N)_k$ currents. The only formal difference appears in the current-current OPE's:

$$I^A_L(z)I^B_L(0) \sim \frac{k\,\delta^{AB}}{8\pi^2\, z^2} + \frac{if^{ABC}}{2\pi\, z}\, I^C_L(0), \tag{B6}$$

with a similar expression for right currents.

The two-loop β-function takes a compact form in terms of the structure constants [63]:

$$\dot{g}_\alpha = -\frac{1}{8\pi}\, C^{\beta\gamma}_\alpha\, g_\beta g_\gamma - \frac{k}{32\pi^2}\, \tilde{C}^{\beta\gamma}_\alpha D^{\mu\nu}_\beta\, g_\gamma g_\mu g_\nu. \tag{B7}$$

The structure constants are calculated to be (we give only the non vanishing entries):

$$\begin{gathered} C^{\parallel,\parallel}_\parallel = -2(N+1), \quad C^{\perp,\perp}_\parallel = -2(N-1), \quad C^{\parallel,\perp}_\perp = C^{\perp,\parallel}_\perp = -2N \\ \tilde{C}^{\parallel,\parallel}_\parallel = 2(N+1), \quad \tilde{C}^{\perp,\perp}_\perp = 2N, \quad \tilde{C}^{\parallel,\perp}_\parallel = 2(N-1), \quad \tilde{C}^{\parallel,\perp}_\perp = 2N, \\ D^{\parallel,\parallel}_\parallel = D^{\perp,\perp}_\perp = 1, \end{gathered} \tag{B8}$$

so that Eq. (B7) gives the two-loop result (B1).

APPENDIX C: $\mathbb{Z}_N$ PARAFERMIONIC CFT

In this Appendix, we discuss the $\mathbb{Z}_N$ parafermionic CFT and present the technical details of the conformal embedding approach of section II which are used in section III for the determination of the phase diagram of model (3).

The parafermionic CFT describes the critical properties of two-dimensional $\mathbb{Z}_N$ generalization of the Ising model. The lattice spin σ_r of such a model takes values: $e^{i2\pi m/N}, m = 0, \ldots, N-1$ and the corresponding Hamiltonian is $\mathbb{Z}_N$ invariant. Using the KW duality symmetry, the spins σ_r are replaced by the dual spins $\mu_{\tilde{r}}$ which belong to the dual lattice and they define a $\tilde{\mathbb{Z}}_N$ symmetry. Two-dimensional $\mathbb{Z}_N$ Ising models display two gapped phases, i.e. the ordered (respectively disordered)

phase where the $\mathbb{Z}_N$ (respectively $\tilde{\mathbb{Z}}_N$) symmetry is spontaneously broken, as well as a quasi-long-range KT phase for prime $N \geq 5$. The critical properties, along the self-dual manifold, are captured by the $\mathbb{Z}_N \times \tilde{\mathbb{Z}}_N$ parafermionic CFT with central charge $c = 2(N-1)/(N+2)$ [36]. In the scaling limit, the conformal fields σ_k and μ_k ($\sigma_k^\dagger = \sigma_{N-k}, \mu_k^\dagger = \mu_{N-k}$) with scaling dimensions $d_k = k(N-k)/N(N+2)$ ($k = 0, \ldots, N-1$) describe the long-distance correlations of σ_r^k and $\mu_{\tilde{r}}^k$. In particular, they carry respectively a $(k, 0)$ and $(0, k)$ charge under the $\mathbb{Z}_N \times \tilde{\mathbb{Z}}_N$ symmetry:

$$\begin{aligned} \sigma_k &\rightarrow e^{i2\pi mk/N} \sigma_k \;\; \text{under} \;\; \mathbb{Z}_N \\ \mu_k &\rightarrow e^{i2\pi mk/N} \mu_k \;\; \text{under} \;\; \tilde{\mathbb{Z}}_N, \end{aligned} \tag{C1}$$

with $m = 0, \ldots, N-1$ and σ_k (respectively μ_k) remains unchanged under the $\tilde{\mathbb{Z}}_N$ (respectively $\mathbb{Z}_N$) symmetry.

The $\mathbb{Z}_N$ parafermionic CFT is generated by the parafermionic currents Ψ_{kL} and Ψ_{kR} ($\Psi_{kL,R}^\dagger = \Psi_{N-kL,R}$) which are holomorphic and antiholomorphic fields with conformal weights: $h_k = \bar{h}_k = k(N-k)/N$. They carry respectively a (k, k) and $(k, -k)$ charge under the $\mathbb{Z}_N \times \tilde{\mathbb{Z}}_N$ symmetry and they transform as follows under the KW duality symmetry:

$$\Psi_{kL} \rightarrow \Psi_{kL}, \Psi_{kR} \rightarrow (-1)^k \, \Psi_{kR}^\dagger. \tag{C2}$$

In addition, these currents satisfy the parafermionic algebra which is defined by the following OPEs [36]:

$$\begin{aligned} \Psi_{kL}(z)\, \Psi_{qL}(\omega) &\sim c_{k,q} (z-\omega)^{-2kq/N}\, \Psi_{k+qL}(\omega) \;\; , \;\; \text{for} \;\; k+q < 4 && \text{(C3)} \\ \Psi_{kL}(z)\, \Psi_{qL}^\dagger(\omega) &\sim c_{k,N-q} (z-\omega)^{-2q(N-k)/N}\, \Psi_{k-qL}(\omega) && \text{(C4)} \\ \Psi_{kL}(z)\, \Psi_{kL}^\dagger(\omega) &\sim (z-\omega)^{-2k(N-k)/N} \left(1 + \frac{2h_k}{c} (z-\omega)^2\, T(\omega)\right), && \text{(C5)} \end{aligned}$$

with similar results for the right parafermionic currents. In Eq. (C5), $T(z)$ denotes the stress-energy tensor of the $\mathbb{Z}_N$ CFT and the numerical coefficients are given by:

$$c_{k,q}^2 = \frac{\Gamma(k+q+1)\,\Gamma(5-k)\,\Gamma(5-q)}{\Gamma(k+1)\,\Gamma(q+1)\,\Gamma(6-k-q)\,\Gamma(5)}. \tag{C6}$$

Apart from these fields, the $\mathbb{Z}_N$ CFT also contains neutral fields, i.e. invariant under the $\mathbb{Z}_N \times \tilde{\mathbb{Z}}_N$ symmetry, ϵ_j ($j = 1, \ldots, [N/2]$) with scaling dimension $D_j = 2j(j+1)/(N+2)$ which are the thermal operators of the theory.

It is well known that the $\mathbb{Z}_N$ CFT can be viewed as a coset (see for instance Refs. [36, 43]): $\mathbb{Z}_N \sim \mathrm{SU}(2)_N/\mathrm{U}(1)_N$ or $\mathbb{Z}_N \sim [\mathrm{SU}(N)_1 \times \mathrm{SU}(N)_1]/\mathrm{SU}(N)_2$. In fact, there is another coset description

of the $\mathbb{Z}_N$ CFT which is crucial for the low-energy approach of section II: $\mathbb{Z}_N \sim \mathrm{SU}(2N)_1/\mathrm{Sp}(2N)_1$. The corresponding character decomposition has been found by Altschuler [56] and reads as follows:

$$\chi\left(\Lambda_k\right) = \sum_{j=0}^{N} B_j^k \chi\left(\lambda_j\right) = \sum_{j=0}^{N} \eta\, c_k^j \chi\left(\lambda_j\right), \tag{C7}$$

where $\chi\left(\Lambda_k\right), k = 0, \ldots, 2N-1$ (respectively $\chi\left(\lambda_j\right)$) are the characters of the conformal tower corresponding to the $\mathrm{SU}(2N)_1$ (respectively $\mathrm{Sp}(2N)_1$) primary fields $\Phi^{(k)}$ (respectively $\phi^{(j)}$) with scaling dimensions: $k(2N-k)/2N$ (respectively $j(2N+2-j)/2(N+2)$). In Eq. (C7), B_j^k are the branching functions of the coset which identify to the level-N string functions c_m^l of the $\mathrm{SU}(2)_N$ current algebra [43, 64]. The latter are related to the branching functions b_k^j of the coset $\mathbb{Z}_N \sim \mathrm{SU}(2)_N/\mathrm{U}(1)_N$: $b_k^j = \eta c_k^j$ (η being the Dedekind function) with the constraint $j \equiv k$ (2) [64].

The identification (C7) and the non-Abelian bosonization rule (9) enable one, in principle, to express any fermionic operators of the $\mathrm{U}(2N)_1$ CFT in terms of a U(1) charge boson field and operators of the $\mathrm{Sp}(2N)_1 \times \mathbb{Z}_N$ CFT. Let us first consider the example of the representation of the $\mathrm{U}(2N)_1$ field: $L_\alpha^\dagger R_\alpha$. Using Eq. (9), we have:

$$L_\alpha^\dagger R_\alpha \sim : \exp\left(i\sqrt{2\pi/N}\Phi_c\right) : \mathrm{Tr}\; g. \tag{C8}$$

The next step of the approach is to use the decomposition (C7) with $k=1$. The $\mathrm{Sp}(2N)_1$ primary field $\phi^{(1)}$, which transforms in the fundamental representation (self-conjugate) of $\mathrm{Sp}(2N)$, will appear as well as a $\mathbb{Z}_N$ field with scaling dimension $(N-1)/N(N+2)$ i.e. σ_1 or μ_1. One way to resolve this ambiguity is to use the parity symmetry P in the continuum limit. Under the parity, $(L,R)_\alpha \rightarrow (R,L)_\alpha$ and $\Phi_{cL,R} \rightarrow -\Phi_{cR,L}$ so that $\mathrm{Tr}\; g \rightarrow \mathrm{Tr}\; g^\dagger$ from Eq. (C8). However, it is known that, under P, σ_k remains invariant while $\mu_k \rightarrow \mu_k^\dagger$ [36]. We thus deduce the following correspondence:

$$L_\alpha^\dagger R_\alpha \sim : \exp\left(i\sqrt{2\pi/N}\Phi_c\right) : \mathrm{Tr}\; g \sim : \exp\left(i\sqrt{2\pi/N}\Phi_c\right) : \mu_1 \mathrm{Tr}\; \phi^{(1)}. \tag{C9}$$

Another interesting operator is the $\mathrm{Sp}(2N)$ singlet $L_\alpha^\dagger \mathcal{J}_{\alpha\beta} R_\beta^\dagger$ field whose representation can be obtained from Eq. (C9) with help of the duality symmetry (27) of section II. Since the latter symmetry corresponds to the Gaussian duality and the KW transformation of the $\mathbb{Z}_N$ Ising model, i.e. $\sigma_k \leftrightarrow \mu_k$, we have:

$$L_\alpha^\dagger \mathcal{J}_{\alpha\beta} R_\beta^\dagger \sim : \exp\left(i\sqrt{2\pi/N}\Theta_c\right) : \sigma_1 \mathrm{Tr}\; \phi^{(1)}, \tag{C10}$$

which is consistent with the parity symmetry since $L_\alpha^\dagger \mathcal{J}_{\alpha\beta} R_\beta^\dagger$ is invariant under P.

A more complicate example is the identification of the operator $M_0 = \epsilon^{\alpha_1 \ldots \alpha_N \beta_1 \ldots \beta_N} R^\dagger_{\alpha_1} \ldots R^\dagger_{\alpha_N} L^\dagger_{\beta_1} \ldots L^\dagger_{\beta_N}$ in the U(1) $\times$ Sp$(2N)_1$ $\times$ $\mathbb{Z}_N$ basis. This field is an SU$(2N)$ singlet and plays a crucial role in the discussion of the physical properties of the spin-F fermionic model (3) since it is the continuum representation of the MS instability. Using the non-Abelian bosonization rules (9), we first find that:

$$M_0 \sim: \exp\left(i\sqrt{2\pi N}\Theta_c\right): \mathrm{Tr}\ \Phi^{(N)}, \tag{C11}$$

$\Phi^{(N)}$ being the SU$(2N)_1$ primary field which transforms into the self-conjugate representation of SU$(2N)$ given by a Young tableau with N boxes and one column. The next step of the approach is to use Eq. (C7) with $k = N$. Let us assume that N is even, i.e. $N = 2n$, so that we find in this case:

$$\chi\left(\Lambda_{2n}\right) = \eta \sum_{p=0}^{n} c_0^{2p} \chi\left(\lambda_{2(n-p)}\right), \tag{C12}$$

where we have used the symmetry property of the level-N string functions: $c^l_m = c^{N-l}_{N-m}$ [64]. It is known that ηc_0^{2p} with $p = 0, \ldots, N/2$ describe the conformal towers corresponding to the p^{th} thermal operators ϵ_p of the $\mathbb{Z}_N$ parafermionic CFT [64]. We thus deduce the following identification of the MS instability for even N:

$$M_0 \sim: \exp\left(i\sqrt{2\pi N}\Theta_c\right): \sum_{p=0}^{N/2} a_p\ \epsilon_{N/2-p}\ \mathrm{Tr}\ \phi^{(2p)}, \tag{C13}$$

a_p being some numerical coefficients which are not important in what follows. When N is odd, a similar approach gives:

$$M_0 \sim: \exp\left(i\sqrt{2\pi N}\Theta_c\right): \sum_{p=0}^{(N-1)/2} a_p\ \epsilon_{(N-1)/2-p}\ \mathrm{Tr}\ \phi^{(2p+1)}. \tag{C14}$$

Finally, we end this Appendix by mentioning an interesting identification of the first parafermionic current $\Psi_{1R,L}$ which generates the algebra (C3, C4, C5). First, we observe that $L^\dagger_\alpha \mathcal{J}_{\alpha\beta} L^\dagger_\beta$ and $R^\dagger_\alpha \mathcal{J}_{\alpha\beta} R^\dagger_\beta$ are chiral Sp$(2N)$ singlets so that only the identity operator in the Sp$(2N)$ sector can occur in their representation. We then find the following identifications:

$$\begin{aligned} L^\dagger_\alpha \mathcal{J}_{\alpha\beta} L^\dagger_\beta &\simeq \frac{\sqrt{N}}{\pi}: \exp\left(i\sqrt{8\pi/N}\ \Phi_{cL}\right): \Psi_{1L} \\ R^\dagger_\alpha \mathcal{J}_{\alpha\beta} R^\dagger_\beta &\simeq \frac{\sqrt{N}}{\pi}: \exp\left(-i\sqrt{8\pi/N}\ \Phi_{cR}\right): \Psi_{1R}, \end{aligned} \tag{C15}$$

which satisfy the parafermionic algebra (C5) for $k = 1$. The representation of the parafermionic current $\Psi_{kR,L}$ with $k \geq 2$ can then also been derived from the correspondence (C15) and the application of the parafermionic algebra (C3,C4).

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